RESAMPLING CONFIDENCE REGIONS AND TEST PROCEDURES FOR SECOND DEGREE STOCHASTIC EFFICIENCY WITH RESPECT TO A FUNCTION

A Dissertation

by

KEITH DANIEL SCHUMANN

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

August 2005

Major Subject: Statistics

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Approved by:

Co-Chairs of Committee,	Randall L. Eubank
	James W. Richardson
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ABSTRACT

Resampling Confidence Regions and Test Procedures

for Second Degree Stochastic Efficiency with Respect to a Function. (August 2005)

Keith Daniel Schumann, B.S., Texas A&M University;

M.S., Texas A&M University

Co-Chairs of Advisory Committee: Dr. Randall L. Eubank
Dr. James W. Richardson

It is often desirable to compare risky investments in the context of economic decision theory. Expected utility analyses are means by which stochastic alternatives can be ranked by re-weighting the probability mass using a decision-making agent's utility function. By maximizing expected utility, an agent seeks to balance expected returns with the inherent risk in each investment alternative. This can be accomplished by ranking prospects based on the certainty equivalent associated with each alternative.

In instances where only a small sample of observed data is available to estimate the underlying distributions of the risky options, reliable inferences are difficult to make. In this process of comparing alternatives, when estimating explicit probability forms or nonparametric densities, the variance of the estimate, in this case the certainty equivalent, is often ignored. Resampling methods allow for estimating dispersion for a statistic when no parametric assumptions are made about the underlying distribution. An objective of this dissertation is to utilize these methods to estimate confidence regions for the sample certainty equivalents of the alternatives over a subset of the parameter space of the utility function.

A second goal of this research is to formalize a testing procedure when dealing with preference ranking with respect to utility. This is largely based on Meyer's work (1977b) developing stochastic dominance with respect to a function and more specific testing procedures outlined by Eubank et. al. (1993). Within this objective, the asymptotic distribution of the test statistic associated with the hypothesis of preference of one risky outcome over another given a sub-set of the utility function parameter space is explored.

I dedicate this work in appreciation and thanksgiving to my family: my loving wife Carrie, my son Luke, my parents, Our Lady, and almighty God the Father, the Son, and the Holy Spirit.

ACKNOWLEDGEMENTS

I would like to express my gratitude to several individuals for their influence and guidance. Dr. James Richardson has been instrumental in facilitating this Ph.D. research by providing resources, time, and camaraderie. I extend my heartfelt appreciation to Dr. Randy Eubank for being the right man at the right time with the right mind to fulfill this project. My appreciation goes out to Dr. Jeff Hart and Dr. Michael Longnecker for their availability and input. Throughout my Ph.D. career, Dr. Longnecker has been a mentor in his example of kindness and willingness to help. In addition, Dr. James Calvin and Dr. Larry Ringer have been sources of good counsel and repartee.

I would also like to thank the Agricultural and Food Policy Center team of professionals as well as the staff of the Texas Risk Management Education Program for their support. A special thanks goes to my colleague and office mate, Paul Feldman, a true friend who has assisted me through his abundant ability in providing multiple vantage points to approach a problem.

It is because of the professionalism, generosity, and instruction of the faculty members with whom I have been affiliated of the Department of Statistics and the Department of Agricultural Economics at Texas A&M University that I have been able to accomplish this work. A special recognition is extended to the Boom Guy for his contributions.

On a personal note, I would like to thank all of my family and friends who have supported me throughout my doctoral endeavors. My cherished and exquisitely gifted wife has maintained our household almost single-handedly during this time. My beloved son Luke was born in the midst of my studies and managed to keep me grounded and eager to face the day, even if I were exhausted, by tugging at me and saying "Here daddy. Get up. Play." My parents, as always, have been pillars of support and encouragement. In addition, I would like to mention those who have been special points of light for me: my sister Dana and my brother-in-law Philip and my soon-to-be born nephew, my grandmother Gertude Schumann, Great Aunt Caroline, Father Mike Sis, Father Keith Koehl, our marriage group and faith community, specifically the Drabek family, my wife's family, Osteen, Phil and Sarah, and our Adopted Aggies.

Trust in the Lord with all your heart,
on your own intelligence rely not;
In all your ways be mindful of him,
and he will make straight your paths.

Proverbs 3: 5–6

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CHAPTER I

INTRODUCTION

Take calculated risks. That is quite different from being rash.

George S. Patton

In the field of economic decision theory, especially in financial applications, a great deal of research has been undertaken to comparatively analyze risky investments. These investments are assumed to be stochastic wealth variables, typically represented by monetary units, rates of return, etc., whose associated probability distributions can be subjectively or empirically inferred. In addition to maximizing returns, a goal in managing a portfolio of risky investments is to properly balance the risk inherent in the individual assets. There is no universal standard for how this is accomplished, but one way to approach this problem is with respect to expected utility theory.

An agent seeking to invest does not necessarily attempt to maximize expected returns because to do so would more than likely lead to investments in the most risky options. For high-risk ventures, large expected returns are a result of potentially large payoffs with low probability of occurrence. Rather, agents are thought to maximize the expected utility of the investment. Utility is a description of the satisfaction an agent garners from a particular action. A utility function is an attempt by which this level of satisfaction is measured, typically as a function of wealth or changes in wealth.

The format and style follow that of *the Journal of the American Statistical Association*.

Under the law of diminishing marginal utility, the utility function operates under some confines and conforms to some general axiomatic behavior (cf. Von Neumann and Morgenstern 1953), but the precise form of the function and the strength of the axioms are subject to an ever-developing debate (see, e.g., Allais and Hagan 1979 and Rabin 2000). A typical restriction on the utility function $u(\cdot)$ is that $u'(x) > 0$ for all levels of wealth x, and an additional restriction that is often imposed is $u''(x) < 0$ for all x. The second restriction defines the preference of the decision-making agent as risk averse.

If assumptions are made about the functional form of utility, this typically leads to a parametric relationship for the random wealth expressed by an investment. The risk aversion of an agent is an index of how much risk is preferred by this individual. In mathematical terms, it is the function r_A that is the negative of the ratio of the second derivative of the utility function to its first derivative: i.e., $r_A(x) = -u''(x)/u'(x)$.

The measure r_A is called a local absolute risk aversion coefficient (Arrow 1965) and according to Pratt (1964), for any utility function u this measure contains "all essential information about u while eliminating everything arbitrary about u". For multi-parametric utility functions or across measures which can no longer be considered local, the complete specification of the utility function is required for analyses, since either it is not possible to elicit the individual parameter values from the risk aversion coefficient or the assumption of relative local aversion does not hold for larger wealth levels. Another indicator of risk aversion is the relative risk aversion coefficient, defined as $r_R(x) = -u''(x)x/u'(x)$.

Using an expected utility framework and by making additional assumptions about the probabilistic form of the investment choices, risky alternatives can be ranked by the ordering of the expected utility of each alternative given a specified parameter space of the utility function. An intuitively appealing means of doing this is by comparing the *certainty equivalents* of risky prospects. The certainty equivalent describes that fixed sum by which an agent would be indifferent between the distribution of the risky prospect and that amount. It is a means by which to describe the entire distribution with one index. Given a utility function $u(\cdot)$, a random wealth variable X, and an initial level of wealth w_0, the certainty equivalent is

$$CE = u^{-1}\{E[u(X + w_0)]\} - w_0,$$

assuming $u^{-1}(\cdot)$ exists and $E[\cdot]$ denotes expectation with respect to the distribution of X, the random wealth. The certainty equivalent employs the agents' utility function to re-weight the possible outcomes of a prospect and express the outcome as a single expectation.

An illustration of the certainty equivalent in relation to the utility function is represented in Figure 1. This shows the relationship between a random wealth, X, the certainty equivalent of the random wealth, CE, with respect to a concave utility function, and the risk premium, π, which is the difference between the certainty equivalent and the expected value of the random variable. This measure can then be compared to the certainty equivalent of other risky alternatives to discern preferences for the agent, where the outcomes that produce the highest certainty equivalents are preferred.

In comparing risky alternatives in this way, many aspects are often overlooked that can impact the interpretation of the results. One of these is comparing the risky outcomes given a single parameterization of the utility function thought to represent the agents' preferences. This is problematic in that it assumes the parameterization, often summarized as local or relative risk aversion, has been estimated correctly and that preferences have not changed over time.

An extension of this type of preference analysis is comparing two sets of para-

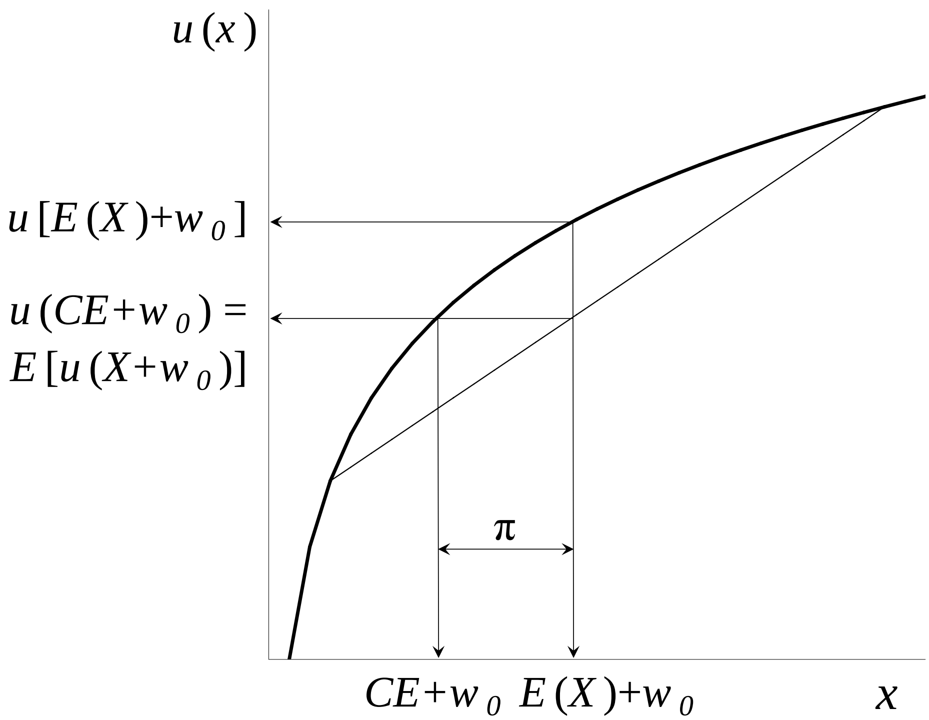


Figure 1. Utility function exhibiting risk aversion and the relative relation of the certainty equivalent (CE), expected value of the random wealth variable ($E(X)$), and the risk premium (π).

meterizations which are meant to represent relevant bounds of risk aversion, thereby creating an efficient set of preferable alternatives. This is a more comprehensive approach in that it attempts to measure a class of risk preferences. But problems can occur if preference changes go undetected within the chosen range, which will cause a member of the efficient set to be unduly eliminated.

Another problem involves not taking the sample properties of the certainty equivalents into account. If the certainty equivalent is an estimate based on sample data from an unknown or estimated distribution, then this statistic alone may not be enough to delineate an efficient set. Indeed, if small samples are taken of potentially highly variable alternatives, the variation of the certainty equivalent may make preference ranking inconclusive. If this variation is ignored, however, rankings may be concluded that are poor reflections of the respective true underlying distributions.

The goal, then, of this dissertation is to attempt to reconcile some of the difficulties of economic decision theory, from the implementation of expected utility, with regard to the sample statistics used in ranking procedures. The first objective will be to use resampling methods to estimate confidence regions for certainty equivalent surfaces given the specific parameterization of an assumed utility function. This will be done across a subspace of the parameters rather than at a single point or boundaries. The second objective will be developing testing procedures for preference ranking given a utility function as a mechanism for choice. This research will attempt to extend the literature by implementing testing procedures with respect to utility functions when the parameter space is largely unknown.

The structure of this dissertation will be as follows. Chapter I has presented the introduction, and the objective of this dissertation. Chapter II will review the literature on expected utility, methods for ranking risky alternatives, and procedures used to test the hypothesis for preference of one alternative over another. Chapter III will develop the methods of deriving the statistics associated with the two primary objectives. Chapter IV presents the properties of the sample certainty equivalents, associated confidence regions, and hypothesis tests under a few common distributional and utility assumptions. An empirical example of the ranking measures, their confidence regions, and test statistics is given in Chapter V. Finally, Chapter VI will provide a summary, the main conclusions of the study, and suggestions for further research. Additional computations are included in the Appendix.

CHAPTER II

LITERATURE REVIEW

2.1 Introduction

The following sections attempt to summarize some of the more relevant works that lay the foundation to the subsequent development of ranking measures for risky alternatives. The first section describes the formalization of expected utility theory, which serves as the framework for later methodologies, as well as some major criticisms of ex and recent applications. The latter section introduces the concept of stochastic dominance and several variations and extensions. In that section, some measures to test hypotheses are explored, specifically one which will be focused on for generalization in the next chapter.

2.2 Utility

Theorization on the measurement of preferential choices in economic decisions began with Bernoulli in the 18$^{\text{th}}$ century. In the context of lotteries and the St. Petersburg Paradox, Bernoulli (1954) hypothesized that rather than attempting to maximize expected payoffs, agents attempt to seek those options which maximize expected utility. Utility became known as an indicator of a person's well-being, which seemed to increase at a decreasing rate relative to prospective wealth. Thus, for a continuous random wealth variable X with distribution function $F(\cdot)$ and a utility function $u(\cdot)$, rather than maximizing $E(X)$, an agent seeks to maximize

$$E[u(X)] = \int u(x)dF(x).$$

The notion of expected utility has been the basis of a significant body of work in economics.

Formal expression of the nature of rational decisions in the context of stochastic outcomes was first presented by von-Neumann and Morgenstern (1953), and later expounded upon, in the form of axioms of preference. The axioms outline the completeness, transitivity, and reflexivity of two lottery alternatives. Further axioms of preference dealt with the independence of combinations of the lotteries and the preference ordering of those lotteries when additional random alternatives were added in a portfolio.

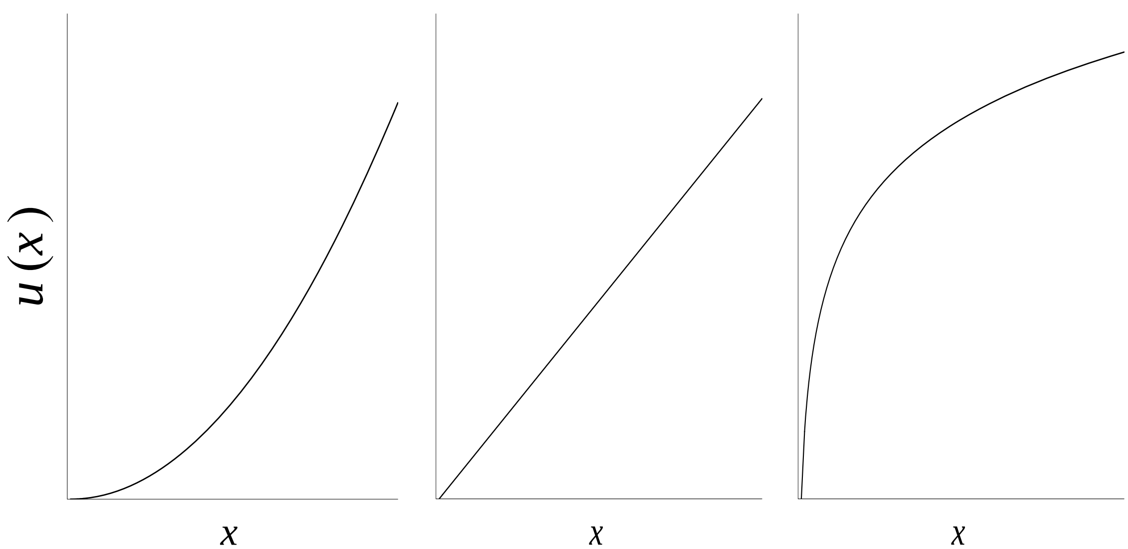
The axiomatic synthesis of preference concepts led to the development of mathematical theoretical concepts of von-Neumann-Morgenstern utility functions. The theorems outline a functional mapping of preferences over a probability measure to a real outcome. These concepts were explored and expounded upon by Friedman and Savage (1948, 1952). The nature of von-Neumann-Morgenstern utility functions was summarized by Pratt (1964) and Arrow (1965) by way of risk aversion.

As previously mentioned, the local risk aversion associated with an increasing utility function is the negative of the ratio of the second derivate of the utility function to the first derivative of the utility function. The derivation is based on the following property of utility

$$u[w_0 + E(X) - \pi(w_0, X)] = E[u(w_0 + X)], \tag{2.1}$$

where w_0 is an initial wealth level and X is a risky asset. The term π is the risk premium, that is defined to be the amount where an agent would be indifferent between the risky asset and the value $E(X)+w_0-\pi$, or $\pi(w_0, X) = E(X)+w_0-CE$. Thus, for risk averse agents, $\pi > 0$. The local absolute risk aversion is then defined through Equation 2.1 by Taking a Taylor expansion around w_0 on both sides of the equation under the assumption that $E(X) = 0$ and that the variance of X is 'relatively' small, or that X is an actuarially neutral risk. Referring back to Figure 1, it can be seen that for risk averse, or concave, utility functions, $\pi > 0$, for linear or risk neutral utility, $\pi = 0$, and for convex utility functions, $\pi < 0$. The sign and magnitude of π are additional indicators of attitudes toward risk, and can be observed as insurance premiums or bets, as the case may be.

The risk aversion coefficient is then a means to measure the concavity of a utility function. This became a way to describe a decision-making agent as locally risk loving, neutral, or averse for a specified wealth level, as the case may be. An illustration of the utility function associated with each classification is given in Figure 2. These indicators are useful in assisting in the determination of risk preferences the agent may make. Means of describing risk attitudes can be elicited by formulations of



(a) Risk Loving (b) Risk Neutral (c) Risk Averse

Figure 2. Utility functions exhibiting strictly (a) risk loving, (b) risk neutral, and (c) risk aversion behavior with respect to wealth (x).

the absolute risk aversion coefficients. The derivative with respect to wealth of the absolute risk aversion coefficient, $r'_A(x)$, indicates changes in absolute amounts of investments. Decreasing absolute risk aversion (DARA) occurs if $r'_A(x) < 0$. DARA implies that as wealth increases, the absolute amount of money invested in the risky prospect increases (such prospects are often called 'normal goods'). If $r'_A(x) = 0$, then the absolute amount of money invested in the risky prospect remains constant as wealth increases. This is called constant absolute risk aversion (CARA). Increasing absolute risk aversion (IARA) occurs when $r'_A(x) > 0$. The amount of money invested in a prospect decreases as wealth increases (such prospects are called 'inferior goods').

Variations of these measures can be derived from the relative risk aversion coefficient, $r_R(x) = r_A(x)x$. The derivative with respect to wealth of $r_R(x)$, $r'_R(x)$, indicates changes in proportional amounts put in risky investments. Decreasing relative risk aversion (DRRA) occurs if $r'_R(x) < 0$ and implies that as wealth increases, the proportional amount of money invested in the risky prospect increases. The remaining relative risk attitudes are constant relative risk aversion (CRRA, $r'_R(x) = 0$) and increasing relative risk aversion (IRRA, $r'_R(x) > 0$).

Expected utility has been established as a means of evaluating relative risk in economics. It has also been the source of criticism, based on one or several of the basic axioms underlying the assumptions of some analyses. One of the earliest critiques came from Allais (Allais and Hagan 1979), whose suppositions and subsequent experimentations appear to show that the independence axiom of expected utility theory is either flawed or overly restrictive. The independence axiom states that if a preference exists between two risky options, and each is combined in the same linear way with another risky option, then the preferences should remain unchanged. The experimentation with pseudo-payoffs used by Allais has been a vehicle for other evaluations of the usefulness of expected utility.

One of the criticisms built on Allais' work developed into an alternative means of evaluation. Kahneman and Tversky (1979) explained the anomalies of expected utility theory by the distinctive psychological phenomena involved in choosing risky ventures. This pattern of analysis, known as prospect theory, dealt in terms of value rather than utility and related decisions to the agent's current assets.

A more recent criticism of the assumption of strict concavity of the utility function came from Rabin (2000). His premise was that this assumption in the framework of expected utility results in absurd assumptions about the risk aversion of an agent when making inferences about modest-scale risks. He presented an alternative means of evaluating preferences, namely the concept of loss aversion, which focuses on changes in wealth rather than the expected total wealth. Albeit plausible in its design, his analyses again relied on experimentations with pseudo-payoffs. Arguments presented to counter this criticism, as in LeRoy (2003), strike at the premise that experimental gambles actually reflect real decisions. When presented with a series of real gambles rather than a one-time pseudo-gamble, it is contended that agents make decisions consistent with those Rabin presents as unlikely.

In more recent works, the use of expected utility has seen revived attention. Abdellaoui (2002) proposed a rank-dependent utility function that holds to von-Neumann-Morgenstern type criteria. He develops the concept of probability risk aversion as a characteristic of a rank-based framework and related it to risk aversion in the typical sense. Extensive empirical applications of this method are yet to develop.

2.3 Stochastic Dominance

Built on the fundamentals of expected utility, stochastic dominance is a more formal method to rank risky alternatives. The development relies heavily on the specific definitions of the relative riskiness of two alternative variables and their relation through mean-preserving spread functions provided in Rothschild and Stiglitz (1970, 1971) and Diamond and Stiglitz (1974). A mean-preserving spread function s is defined with relation to two random variables X and Y, with distribution functions $F(\cdot)$ and $G(\cdot)$, respectively, by taking $G = F + \delta$, where

$$\delta(x) = \int_{-\infty}^{x} s(t)dt,$$
$$\delta(x) \geq 0 \;\forall\; x \leq z \text{ for some } z \in Range(X),$$
$$\delta(x) > 0 \;\text{ for some } x,$$
$$\int s(x)dx = 0,$$
$$\int xs(x)dx = 0,$$

and $F + \delta$ is non-decreasing. This spread function re-weights the probability mass of a random variable from the center to the tails. For two random variables with equal means, if the distribution function of one can be arrived at by adding a mean-preserving spread function a finite number of times to the distribution function of the second, then an assessment of relative riskiness can be made. When $G = F + \delta$, this is defined as first degree (or order) stochastic dominance (FSD), even in the case that the random variables have the same mean.

FSD is just another way of saying one random alternative is stochastically larger than a second. This can be understood if δ is considered to be a noise term. Consider two risky opportunities that are investment possibilities for an agent. These random variables are valued by monetary net returns or rates of return and have common support. For the random variables, X and Y, with corresponding distribution functions $F(\cdot)$ and $G(\cdot)$, $X\ FSD\ Y$ (or $F\ FSD\ G$) means that

$$F(x) \leq G(x) \;\forall\; x \tag{2.2}$$

with at least one strict inequality over the common support of X and Y. An illustration is given in Figure 3.

Hadar and Russell (1969) as well as Hanoch and Levy (1969) are commonly attributed as outlining the concepts of first as well as second degree stochastic dominance. These authors also criticize the use of mean-variance type analyses as general methods to evaluate risky options.

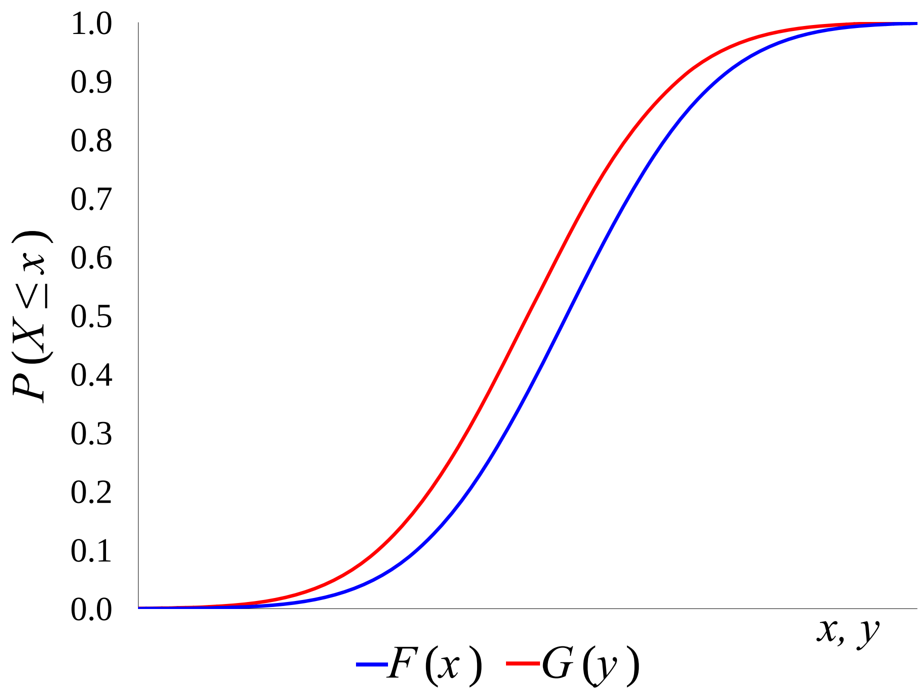


Figure 3. *Cumulative distribution functions (CDFs) of two alternative investments, X and Y such that X first degree stochastically dominates Y.*

Again, for two risky alternatives X and Y with corresponding distribution functions $F(\cdot)$ and $G(\cdot)$, X second degree (or order) dominates Y (denoted X SSD Y) means that

$$\int_{-\infty}^{x} [G(t) - F(t)]dt \geq 0$$

over the common support of X and Y for all x values. This principle assumes an agent is risk averse and prefers more to less, or prefers to maximize the area between the curves if they cross. Clearly FSD implies SSD. This method opened up further criticisms for mean-variance approaches to ranking risky alternatives, where both of the aforementioned sets of authors argue that the first two moments of a distribution are not generally sufficient to rank preferences. Since the first and second moments do not, in general, describe how or if distributions cross, the specific focus of risk analysis, i.e., describing what occurs in the tails of the distributions, can be overlooked. This can be seen in Figure 4.

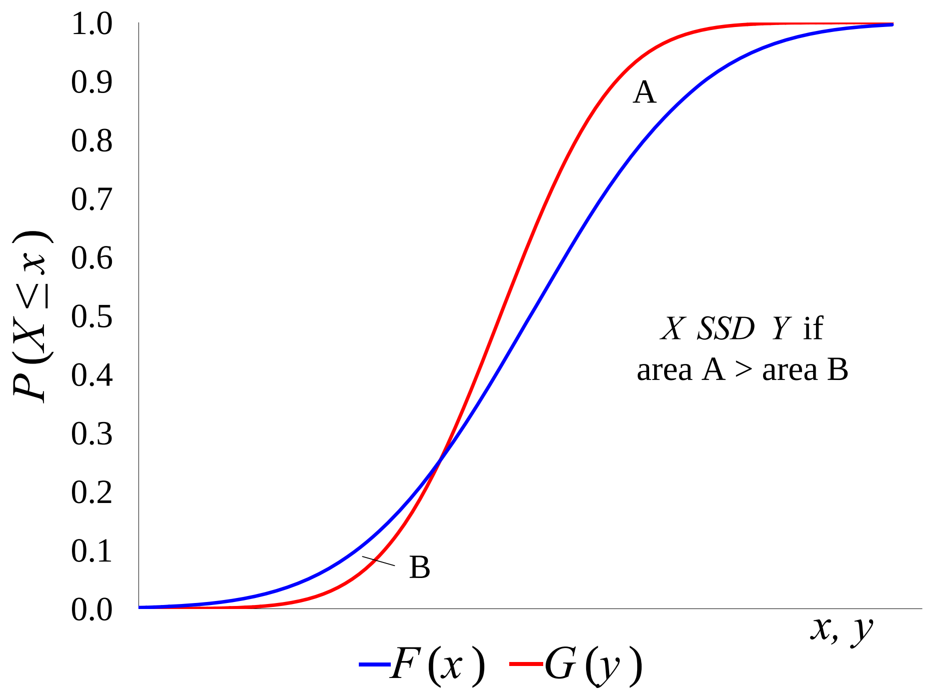


Figure 4. Cumulative distribution functions (CDFs) of two alternative investments, X and Y such that X second degree stochastically dominates Y.

Meyer (1975) laid the foundation for another measure of stochastic dominance, specifically stochastic dominance with respect to a function. Assuming a strictly increasing utility function, he used a type of the aforementioned spread function, or a function which re-weights a cumulative distribution function from the center to the tails, to prove, for a restricted support random variable that was standardized to have a range from 0 to 1, that

$$\int_0^y [G(t) - F(t)]dr(t) \geq 0$$

for all $y \in [0, 1]$ given an increasing, twice differentiable function $r(\cdot)$ if and only if $G(\cdot)$ is at least as risky as $F(\cdot)$. The riskiness of a random variable is understood with respect to the utility preference or decision-making mechanism an agent employs. All else being equal, a more risky prospect is one that has a higher relative variability. For an agent who is averse to risk, i.e., a person who chooses to insure against risk, minimizing risk or variability in the lower tail of a distribution is the primary concern. In this phraseology, risk is defined in terms of the absolute risk aversion coefficient,

$$r_A(x) = -u''(x)/u'(x),$$

where u is the utility function. Thus, given two random alternatives to choose from, an agent with risk aversion evaluated as r_A will prefer the alternative that maximizes $E[u(X)]$, or the less risky alternative. This method was shown to be synonymous with minimizing the maximum loss for preferences. The procedure holds in general for random variables with unbounded ranges.

Meyer (1975) formalized the concept of second degree stochastic dominance with respect to a function $k(\cdot|\theta)$, or simply $k(\cdot)$, (SSD(k_θ) or SSD(k)) with an application (Meyer 1977a) and theoretical development (Meyer 1977b). For two restricted range random variables, that is, random variables transformed to the [0,1] interval, in particular, and their corresponding distribution function, $F\ SSD(k)\ G$ if and only if

$$\int_0^y [G(t) - F(t)]dk(t) \geq 0$$

for $y \in [0,1]$ and a function $k(\cdot|\theta)$ with a given value of θ. Thus, this is a necessary and sufficient condition for $F(\cdot)$ to be preferred to or indifferent to $G(\cdot)$ by all agents with a utility function $u(\cdot)$ that exhibits equal or more risk aversion, or concavity, than the function $k(\cdot)$. This is with respect to a bounded, specified range of risk aversion of the utility function in question, which in application results in examination at the boundaries of risk aversion.

The SSD(k) ordering result is unique if the distribution functions cross a finite number of times. Thus, SSD is a special case of SSD(k) where $k(x) = x$. The benefit of this method is that $k(\cdot)$ is an arbitrary, twice differentiable function, so skepticism about the monotonicity or strict concavity of a utility function in the context of expected utility theory can be moderated by the application of this principle. Thus, the utility function plays an explicit part in the evaluation of stochastic dominance, rather than acting as an implied criterion in typical first and second degree dominance evaluation procedures. An illustration of the functions evaluated by integration in the SSD(k) procedure is given in Figure 5.

Another generalization of stochastic dominance with respect to a function is called stochastic efficiency with respect to a function (SERF) (see Hardaker, et. al. 2004). This method seeks to examine the dominance of one alternative over another given a continuous range of risk aversion as originally specified by Meyer (1977b). The problem with the application of stochastic dominance with respect to a function is that order preferences were evaluated only at the boundaries of the specified risk aversion range. For two alternatives with distribution functions that cross multiple times, this could lead to misspecifications of the efficient, or preferred, set of alternatives over that range. The SERF method evaluates the certainty equivalent of each alternative over the relevant parameter space of the utility function. A SERF chart comparing two normally distributed alternatives is given in Figure 6.

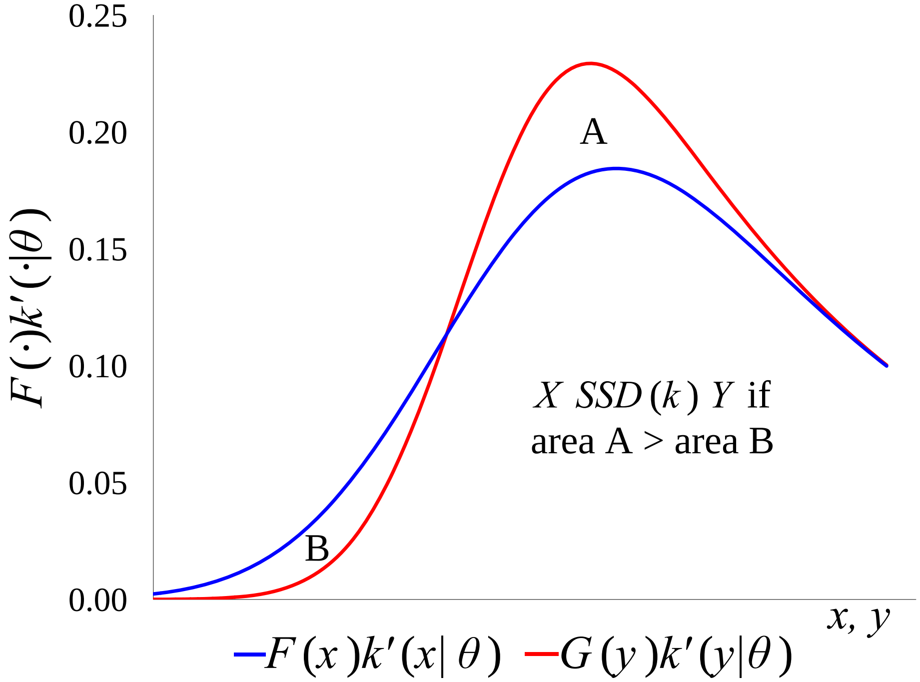


Figure 5. Utility-weighted distribution functions of two normally distributed alternative investments, X and Y such that X second degree stochastically dominates Y with respect to the utility function $k(x|\theta) = \delta - e^{-\theta x}$. In this case, $\theta = r_A = 0.010$.

Several methods have been proposed to develop statistical hypothesis tests for ranking one random alternative over another, or placing alternatives in an efficient set of preferred alternatives, especially given nonparametric assumptions on the underlying distributions. Anderson (1996) proposed nonparametric tests for first, second, and third degree stochastic dominance criteria based on analogs of Pearson goodness of fit tests. These tests were shown to be comparable in size and power to generalized Lorenz curve methods (as in Bishop, et. al. 1989) for comparing distributional differences for wealth.

In addition to mean-variance methods and stochastic dominance criteria, efforts to link these methods and related numerical procedures have been undertaken.

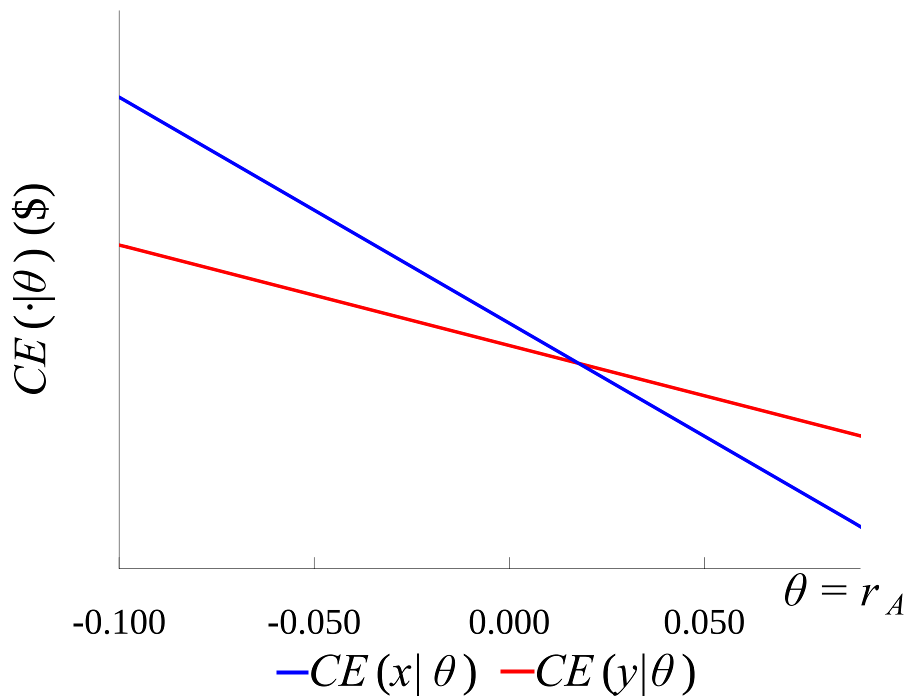


Figure 6. *Certainty equivalent lines of two normally distributed alternative investments, X and Y such that X exhibits stochastic efficiency over Y with respect to the utility function $k(x|\theta) = \delta - e^{-\theta x}$ over the parameter space $\theta \in (-\infty, 0.018]$ and Y exhibits stochastic efficiency over X over the parameter space $\theta \in (0.018, \infty)$.*

Yitzhaki (1982) introduced methods to incorporate Gini's mean difference (GMD) in the analysis of preference ranking, and he elaborated this work by utilizing resampling methods to calculate the variance of estimates of this type (Yitzhaki 1991). A comparison of mean-GMD analyses and stochastic dominance results as well as an application in agricultural commodities was illustrated in McDonald, et. al. (1997). Here it was suggested that mean-GMD methods of analyzing the efficient set of preferred alternatives had superior properties over stochastic dominance methods. Shalit and Yitzhaki (1994) introduced the concept of marginal conditional stochastic dominance (MCSD) and Seiler (2001) proposed a nonparametric test for this preference ranking method that will be discussed further in the following chapter.

Another work in the area of hypothesis testing for preferences was Eubank, Schechtman, and Yitzhaki (1993), hereafter referred to as ESY. ESY described a nonparametric method to test for second degree stochastic dominance. This work was a generalization of a methodology developed by Deshpande and Singh (1985), where the distribution of one of the risky prospects was assumed to be known for testing purposes. The asymptotic properties of the SSD test statistic, including asymptotic power, were elicited in ESY. The large sample variance of the statistic had a form that was cumbersome from an estimation perspective, so a resampling method to approximate the variance was suggested.

A brief overview of the area of economic risk analysis has been given. Expected utility theory has been explored as a means to evaluate preferences when an agent has to choose between random outcomes. In this framework, several methods have been outlined to compare risky alternatives which fall under the category of stochastic dominance analyses. In particular, stochastic dominance with respect to a function has been shown to be a method of preference ranking given the explicit utility function assumption used to model decision-making behavior. Some empirical methods to test particular rankings under differing stochastic dominance procedures have been touched upon. The remaining chapters will explore some generalizations of the empirical testing procedures outlined above, in particular that of SSD detailed in Eubank, Schechtman, and Yitzhaki (1993).

CHAPTER III

METHODS

3.1 Introduction

The remaining sections will proceed assuming the validity of the expected utility framework as a means for economic analysis of risky alternatives. Central to this assumption is that, despite criticisms enumerated in the previous section, the existence of the certainty equivalent is not only intuitive but demonstrable in observable transactions. Moreover, given that a marketplace exists for insurance and lotteries, then a salable value expressed at levels other than the respective expected values indicates a preference weighting of possible outcomes from the potential purchaser. This preference weighting can be achieved by the use of a utility function. The exact form of this utility function is not herein addressed. Example utility forms will be considered, but the generalization of the methodology of empirically evaluating preference ranking using utility functions will serve to advance the literature in this field.

It should be noted that the methodology to follow is not inherently limited to standard utility analyses. Changes in wealth could be easily substituted for the actual wealth outcomes used. Additionally, functional forms utilized in the context of prospect theoretics could also be applied in these empirical methods.

The following sections will develop an empirical methodology for comparing risky alternatives with respect to preferences based on an agent's utility. The assumptions of the problem framework will be described in the first section. The next section will illustrate the method to estimate confidence regions for the certainty equivalents given different parameterizations of the utility function. The final section explains the development of the test statistic used to test for stochastic dominance with respect to a function.

3.2 Problem Framework for Preference Ranking

Consider an agent with an initial level of wealth who wishes to invest in a subset of a finite number of risky investment alternatives. The agent's initial wealth will be labeled w_0, and J risky opportunities, $X_j, j = 1, \ldots, J$, will be considered for investment. It is assumed that the agent (or agents) has a single, unknown utility function for decision-making involving wealth, measured with a proxy utility function $k(\cdot|\theta)$, where the unknown parameter set θ describes the specifications of that utility function.

The risk proclivity specific to the utility function is determined by θ as it relates to the wealth variable X. In this methodology, the immediate goal will not be to estimate values for the parameter θ. The value for θ will be assumed to be fixed and given. The class of utility functions to be considered will be increasing and twice differentiable with respect to the first argument. This ensures that the local risk aversion coefficient exists for all X.

Random samples of size $n_1 = \cdots = n_J = n$ from each of the J prospective investments will be considered as a basis for estimating the distribution functions of each X_j. Typically, these are time-indexed observations of the historical returns of the investments. It should be noted that appropriate care should be taken to model any significant time-oriented impacts, i.e., deflate the deterministic trends in the observations, so that assumptions of independence can be made. For period $i, i = 1, \ldots, n_J$, the observations on the investments will be denoted as $x_{1i}, \ldots, x_{Ji}$. The investment prospects can be arranged in a portfolio based on the percentage of wealth assigned to each.

Given the previous conditions, the expected utility of each investment is defined as

$$Eu(X_j|\theta) = \int_{\chi_j} u(t|\theta)dF_j(t), \tag{3.1}$$

where $F_j(\cdot)$ is the cumulative distribution function of the investment j over χ_j, the support of X_j. The certainty equivalent for investment alternative j based on the specific utility function, $CE_j[u(\cdot|\theta)]$ (or simply $CE_j(\theta)$), is defined by the relationship

$$u[CE_j(\theta) + w_0] = E[u(X_j + w_0|\theta)], \tag{3.2}$$

or

$$CE_j(\theta) = u^{-1}\{E[u(X_j + w_0|\theta)]\} - w_0. \tag{3.3}$$

This is the utility weighted value representing that amount the agent would be indifferent between given the random prospect and a fixed amount.

Since the distribution functions of the alternatives are assumed to be unknown, the empirical distribution function (EDF), $F_{jn}(\cdot)$, will be considered as an estimate of the true distribution $F_j(\cdot)$. The EDF will be defined as

$$F_{jn}(x) = \frac{1}{n}\sum_{i=1}^{n} I_{(-\infty,x_i]}(x), \tag{3.4}$$

where $I_A(z)$ is the indicator function which takes on a value of one if $z \in A$ and zero otherwise. Given this, the sample certainty equivalent associated with alternative j, $\widehat{CE}_j(\theta)$, is then defined as

$$\begin{aligned} \widehat{CE}_j(\theta) &= u^{-1}\left\{\int_{\chi} u(t + w_0|\theta)dF_{jn}(t)\right\} - w_0 &(3.5)\\ &= u^{-1}\left\{n^{-1}\sum_{i}^{n} u(x_{ij} + w_0|\theta)\right\} - w_0\ . &(3.6)\end{aligned}$$

For ranking purposes, for the set Ω_J of all known J alternatives in question, the sample efficient set, Ω_E, is that subset of the J alternatives which are preferable over

θ, i.e., alternatives that belong to the surface

$$\arg\max_{J} \widehat{CE}_j(\theta) \text{ for } \theta \in \Theta \ . \tag{3.7}$$

Note that since the certainty equivalent is measured in terms of the units of the random wealth X_j, it also lies in the range of the random variable X_j, and therefore members of the efficient set do as well.

Figure 7 is an example of allocating alternatives to an efficient set. Assuming a von-Neumann Morgenstern type of utility function, let θ be the single utility parameter and an index of risk aversion such that increasing θ increases risk aversion. For the four alternatives $X_j, j = 1, \ldots, 4$, if it is assumed that the depicted certainty equivalents represent the true values, all but X_4 could be allocated to the efficient set, Ω_E, depending on θ. Specifically, $X_2 \in \Omega_E$ over $\Theta = (-\infty, -0.05855]$ and $\Theta = (0.04805, \infty)$, $X_3 \in \Omega_E$ over $\Theta = (-0.05855, 0.01586]$, and $X_1 \in \Omega_E$ over $\Theta = (0.01586, 0.04805]$. For the entire range, $\Theta = (-\infty, \infty)$, $X_1, X_2, X_3 \in \Omega_E$.

The levels and magnitude of individual θ values have varying interpretations and associations to risk aversion. It is best to consider them in general categories of extremely risk loving, moderately risk loving, risk neutral, moderately risk averse, and extremely risk averse, or some similar division. Therefore, X_3 is preferred for risk neutral agents.

Note that X_2 is preferred for all extremely risk loving agents as well as all extremely risk averse agents. This can be explained by the agent's particular focus on one tail of the respective distribution or another. Assuming a restricted support, this particular option has the largest minimum value relative to the others as well as the largest maximum value.

In the context of a SERF analysis, the relevant parameter space based on the sample efficient set, Θ_E, is that subset of Θ where all preference changes between

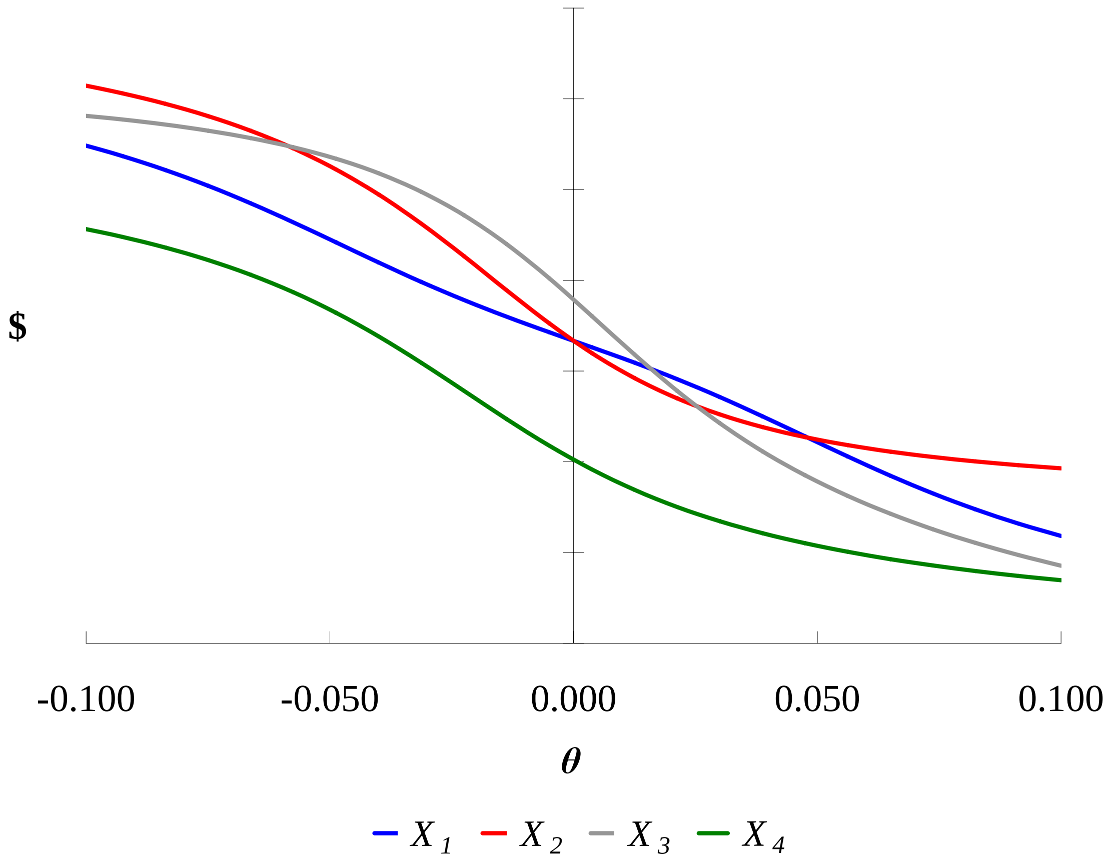


Figure 7. *Stochastic efficiency with respect to a function (SERF) for four alternative investments, $X_i, i = 1, \ldots, 4$. The parameter of a concave utility function, θ, represents the level of risk aversion.*

alternatives have taken place. For many utility functions, preferences cease to change at the point where a predominance of the utility weighting tends to the sample minimum or maximum of each alternative. Thus, for several common utility functions, Θ_E is bounded by $\inf_J x_{j(1)}$ and $\sup_J x_{j(n)}$.

3.3 Confidence Regions

Given that the previous development assumes nonparametric distribution functions, confidence regions for each certainty equivalent surface can be generated using resampling methods. One method involves using resampling of the data to develop a bootstrap representation of the distribution of the certainty equivalent surface. The second method involves using jackknife procedures to estimate the variance of the certainty equivalent surface.

For the first resampling procedure, B bootstrap samples of the original data series are generated. The b^{th} bootstrap sample of the j^{th} alternative is denoted as $x_j^{(b)}$ and the corresponding sample certainty equivalent, $\widetilde{CE}_j^{(b)}(\theta)$, is calculated as

$$\widetilde{CE}_j^{(b)}(\theta) = u^{-1}\left\{n^{-1}\sum_{i}^{n} u(x_{ji}^{(b)} + w_0|\theta)\right\} - w_0 \ . \tag{3.8}$$

These values are then used to develop bootstrap confidence regions for the efficient set given a specified parameter space. The sample quantiles associated with the desired confidence level are used as the outer boundaries of the region. This method produces a non-unique, asymptotic confidence region for continuous support random return variables. An overview of this procedure can be found in Davison and Hinkley (1997).

The resulting confidence regions have the characteristic of being skewed such that the probability mass shifts toward the sample extremes, depending on the parameterization of the utility function. For subsets that represent risk averse preferences, the mass shifts toward the sample minimum. For those subsets that represent risk loving behavior, the mass shifts toward the sample maximum.

There are several potential drawbacks of using the bootstrap method for estimating confidence regions for certainty equivalents. A small sample size is a significant concern. A small sample typically does not adequately represent the true variability of the random variable. In addition, a small sample from a continuous distribution can be somewhat discretized by bootstrap resampling.

The properties of the underlying distribution are another concern in bootstrap sampling. Since estimating the mean and variance of the certainty equivalent is the goal, it is crucial for these moments to exist for the bootstrap sample to be an effective method of estimation. If the supports of the underlying distributions of the alternatives are unrestricted or the distributions are heavy-tailed, then the sample extreme values act to restrict the sample bootstrap confidence region. This is of particular concern when the utility parameter θ is valued such that the associated indices of risk aversion are highly risk loving or risk averse. In these cases, the certainty equivalent approaches either an extreme value of the distribution or an outlier, as the case may be. These problems in the application of bootstrap sampling are outlined in Chernick (1999).

As a closely related alternative to bootstrap sampling, another resampling procedure involves using jackknife estimates of the variance of the certainty equivalent for confidence region estimation. A jackknife estimate of a statistic $t = t(X_1, \ldots, X_n)$ is based on calculating a new statistic $t_{(i)}$ by leaving out one observation, or $t_{(i)} = t(X_1, X_2, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n)$. Then, for $i = 1, \ldots, n$, the statistics are averaged and result in $\hat{t}_{(\cdot)}$. A sample standard error can then be calculated as $\widehat{SE(t)} = \sqrt{\frac{n-1}{n} \sum_{i=1}^{n} (t_{(i)} - \hat{t}_{(\cdot)})^2}$. For the certainty equivalent, this is

$$\widehat{SE}\left[CE_j(\cdot|\theta)\right] = \sqrt{\frac{n-1}{n} \sum_{i=1}^{n} (CE_{j(i)} - \widehat{CE}_{j(\cdot)})^2}. \tag{3.9}$$

The sample certainty equivalent, for most locally monotonic utility functions, is a one-to-one function of an average of a functional transformation of the sample data. For an independent sample, the sample certainty equivalent has an asymptotic normal distribution as a result of applying the central limit theorem and the delta method. A confidence region can then be created by using the fact that both the statistic and the jackknife standard error, assuming it is consistent, to approach the true mean certainty equivalent and the true standard deviation of the certainty equivalent in the limit, respectively, as $n \to \infty$. Thus, an asymptotic $100(1 - \alpha)\%$ confidence region can be developed using normal quantiles.

3.4 Hypothesis Test

The second objective of this research involves developing a formal hypothesis testing procedure in the context of stochastic efficiency with respect to a function. This entails incorporating the progressive works of Meyer (1975, 1977a, and 1977b) on explicit stochastic dominance with respect to a function and generalizing it using the procedures outlined in ESY.

Referring again to the definition of second degree stochastic dominance with respect to a function, where $F\ SSD(k_\theta)\ G$ if and only if

$$\int_{-\infty}^{x} [G(t) - F(t)]dk(t) \geq 0$$

for all x, a function $k(\cdot|\theta)$, and a given value of θ, a formal testing procedure can be elicited from the methods in ESY. In this procedure, the hypotheses are expressed as

$$H_0 \ : \ F = G \tag{3.10}$$

$$\text{and} \qquad H_1 \ : \ F\ SSD\ G\ .$$

Letting

$$d_{F,G}(x) = \int_{-\infty}^{x} [G(t) - F(t)]dt \text{ and} \tag{3.11}$$

$$D_{F,G} = \frac{1}{2}\left[\int_{-\infty}^{\infty} d_{F,G}(x)dG(x) + \int_{-\infty}^{\infty} d_{F,G}(y)dF(y)\right]\ , \tag{3.12}$$

the hypothesis is restated as

$$H_0 \ : \ D_{F,G} = 0 \tag{3.13}$$

$$H_1 \ : \ D_{F,G} < 0.$$

The sample test statistic based on two independent samples, $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$, is then

$$D_{n,m} = \frac{1}{2}\left\{\left[\bar{Y} - \frac{1}{2}GMD(Y)\right] - \left[\bar{X} - \frac{1}{2}GMD(X)\right]\right\} \ , \tag{3.14}$$

where $GMD(\cdot)$ is the Gini's mean difference of a sample, defined as

$$GMD(X) = \frac{1}{\binom{n}{2}} \sum \sum_{i<j} |X_i - X_j|. \tag{3.15}$$

The statistic $D_{n,m}$ has an asymptotic normal distribution with variance estimated, for example, by jackknife resampling methods.

It will be shown that this statistic can be generalized for this testing procedure to allow for hypotheses of the type:

$$\begin{aligned} H_0 \ &: \ F = G \ \forall \ \ \theta \in \Theta_1 \qquad (3.16) \\ H_1 \ &: \ F \ SSD(k_\theta) \ G \ \forall \ \ \theta \in \Theta_1 \ , \end{aligned}$$

where Θ_1 is the given parameter subset of interest of the given utility function $k(\cdot|\theta)$. A more general form of the statistic $D_{n,m}$ will be developed, along with the resampled variance estimate of the statistic and the corresponding asymptotic properties. This will then be used to develop confidence regions for the statistic given a subset of the utility function parameter space.

A supposition regarding the nature of the general form of the test statistic is that it might simply be based on the transformations into utility measures and remains structurally similar. Given that the utility functions in the general class in question are one-to-one and assuming these functions are locally monotonic, it is straightforward to conjecture that transforming the variables to utility measures and then using these as the basis for the test procedure in ESY, the desired goal might be achieved. The following development will confirm this supposition.

Using the previous definitions for SSD(k) and the general notation of ESY, the generalized forms of the dominance measures become

$$\begin{aligned} d_{F,G}(x|\theta) &= \int_{-\infty}^{x} [G(t) - F(t)]dk(t|\theta) \text{ and} && (3.17) \\ D_{F,G}(\theta) &= \frac{1}{2}\left[\int_{-\infty}^{\infty} d_{F,G}(x|\theta)dG(x) + \int_{-\infty}^{\infty} d_{F,G}(y|\theta)dF(y)\right], && (3.18) \end{aligned}$$

where $F_n(\cdot)$ and $G_m(\cdot)$ are the EDFs associated with X and Y, respectively, as previously defined. Thus the corresponding sample statistics are

$$\begin{aligned} d_{n,m}(x|\theta) &= \int_{-\infty}^{x} [G_m(t) - F_n(t)]dk(t|\theta) && (3.19) \\ \text{and} \qquad D_{n,m}(\theta) &= \frac{1}{2}\left[\int_{-\infty}^{\infty} d_{n,m}(x|\theta)dG_m(x) + \int_{-\infty}^{\infty} d_{n,m}(y|\theta)dF_n(y)\right]. && (3.20) \end{aligned}$$

The sample form of the generalized statistic $D_{n,m}(\theta)$ is based on the following two theorems:

Theorem 1. Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$ be two independent samples of size n and m from distribution functions F and G, respectively, and let $k(\cdot|\theta)$ be an increasing utility function that is twice differentiable with respect to its first argument. Then

$$\begin{aligned} D_{n,m}(\theta) &= \frac{1}{2}\left[\frac{1}{n}\sum_{i=1}^{n} W_i - \frac{1}{m}\sum_{j=1}^{m} T_j\right] \\ &+ \frac{1}{2}\left[\int_{-\infty}^{\infty} F_n(x)\left[1 - F_n(x)\right]dk(x|\theta) - \int_{-\infty}^{\infty} G_m(y)\left[1 - G_m(y)\right]dk(y|\theta)\right], \end{aligned}$$

where $W_i = \int_{x_i}^{\infty} [k(t|\theta) - k(x_i|\theta)]dG_m(t)$ and $T_j = \int_{y_j}^{\infty} [k(t|\theta) - k(y_j|\theta)]dF_n(t)$.

Theorem 2. Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$ be two independent samples of size n and m from distribution functions F and G, respectively, and let $k(\cdot|\theta)$ be an increasing utility function that is twice differentiable with respect to its first argument.

Then

$$D_{n,m}(\theta) = \frac{1}{2}\left\{\overline{k_\theta(Y)} - \overline{k_\theta(X)} + \frac{1}{2}GMD[k_\theta(X)] - \frac{1}{2}GMD[k_\theta(Y)]\right\} \ ,$$

where $\overline{k_\theta(X)}$ is the average of the n transformed random variables $k(X_i|\theta)$ and $GMD[k_\theta(X)]$ is the Gini's mean difference of the transformed variables $k(X_i|\theta)$, and $\overline{k_\theta(Y)}$ and $GMD[k_\theta(Y)]$ are defined similarly.

The proofs of Theorems 1 and 2 are given in Appendix A. Based on the theorems, it can be seen that the generalized form of the statistic $D_{n,m}$ is arrived at by simply transforming the original data to a utility measure via the utility function and then applying the hypothesis testing framework outlined in ESY.

Given this, the asymptotic properties of the test statistic developed in ESY hold for the generalized form based on the aforementioned transformation procedure. Letting $N = n + m$, the sum of the two sample sizes, the test statistic limiting distributions can be developed under the local alternatives of the form

$$G(x) = F(x) + \delta(x)\sqrt{N}. \tag{3.21}$$

Note that the second term on the right hand side of Equation 3.21 is just a type of spread function as defined previously. Using the empirical distributions, δ can be estimated as

$$\hat{\delta}_N(x) = \frac{G_m(x) - F_n(x)}{\sqrt{N}}. \tag{3.22}$$

Under some conditions[1] on the spread function δ, which hold in general for the distribution functions of monetary returns which cross a finite number of times, the

[1]For the local alternative relation, $G(x) = F(x) + \delta(x)\sqrt{N}$, δ must be a function such that $\int_{-\infty}^{x} \delta(y)dy \geq 0 \ \forall \ x$ and $\int_{-\infty}^{x} \delta(y)dy > 0$ for some x. Under the generalized version, to conform with ESY Theorem 3.1, then $E[V^3] < \infty$, δ is absolutely continuous, $\int |\frac{\partial}{\partial t}\delta(t)|dt < \infty$, $\int |t|^3 |\frac{\partial}{\partial t}\delta(t)|dt < \infty$, $\int \delta^2(t)dt < \infty$, and $0 \leq \delta \leq 1 - F$. See ESY for further details.

null hypothesis implies that $\delta = 0$ and the distributions are the same. In this case, transforming the distributions by utility re-weighting will have no impact on the outcome of the test. So, based on Theorem 3.1 of ESY, and letting $V = k(X)$ and $W = k(Y)$, it can be shown that

$$\sqrt{N} D_{n,m}(\theta) \xrightarrow{\mathcal{D}} N(\mu_{D_{X,Y}(\theta)}, 4\zeta/\lambda(1-\lambda)), \tag{3.23}$$

where

$$\mu_{D_{X,Y}(\theta)} = -2 \int [1 - F_V(t)]\, \delta_V(t) d(t), \tag{3.24}$$

$$\zeta = \frac{1}{4} Var \left[V - \int |V - z| dF_V(z) \right]. \tag{3.25}$$

Here λ is defined as

$$\lambda = \lim_{N \to \infty} \frac{n}{N}$$

and it is assumed that $0 < \lambda < 1$. Using the estimate for δ, the mean of the statistic can be estimated with

$$\hat{\mu}_{D_{n,m}(\theta)} = \frac{-2}{\sqrt{N}} \left[\frac{1}{nm} \sum_{i=1}^{n} \sum_{j=1}^{m} (v_i - w_j) I_{(w_j < v_i)} - \frac{1}{2} GMD(v) \right] \tag{3.26}$$

where $I_{(\cdot)}$ is an indicator function and $GMD(\cdot)$ is Gini's mean difference (see Appendix A for the derivation of $\hat{\mu}_{D_{n,m}(\theta)}$).

Under the null hypothesis, the measure δ is equal to zero and so $\mu_{D_{X,Y}(\theta)}$ is as well. The result is that

$$N^{-1/2} D_{n,m}(\theta) / (2\sqrt{\zeta/nm}) \xrightarrow{\mathcal{D}} N(0, 1). \tag{3.27}$$

As described in ESY[2], the variance term for $D_{n,m}$ can be estimated using a jackknife estimator since $D_{n,m}$ is a function of U-statistics (cf. Serfling, 1980, Chapter 5). Thus,

[2] In the original work, the rate of convergence of the statistic was inaccurately presented as $N^{3/2}$. The proper rate of convergence is $N^{-1/2}$.

H_0 is rejected if

$$N^{-1/2}D_{n,m}(\theta)/(2\sqrt{\zeta/nm}) < -z_\alpha, \tag{3.28}$$

for large N, where $-z_\alpha$ is the lower quantile associated with the $100(1-\alpha)\%$ percentile of the standard normal distribution. The sample form of the statistic based on replacing the variance term with the jackknife estimate of variance is defined as

$$T_N(\theta) = D_{n,m}(\theta)/\widehat{SE}_{D_{n,m}(\theta)}, \tag{3.29}$$

where $\widehat{SE}_{D_{n,m}(\theta)}$ is defined similarly to that given in Equation 3.9.

Calculations for the statistic $D_{n,m}$ can be made for $n \neq m$ and the order of the sample data is not important. The formulations for the standard error of this statistic, especially when employing the jackknife method for estimation, are more specific. If the sample sizes are not equal, some interpolation may be necessary to standardize the samples to the same size, preferably the larger sample size. Additionally, if the samples are independent within sample but dependencies exist across the variables within an observation, maintaining these dependencies for the jackknife variance estimation procedure is essential for a proper representation of the true variance of the statistic.

CHAPTER IV

EXAMPLE APPLICATIONS

4.1 Introduction

This chapter will develop some of the concepts of stochastic efficiency with respect to a function, especially focusing on particular cases of known distributions and forms of utility. Section 4.2 will demonstrate some of the properties of the certainty equivalent under varying utility and distributional assumptions. The point of this section is to introduce some assumptions commonly made in expected utility preference analyses and demonstrate how difficult it is, even under the assumptions, to derive analytical expressions for the variance (and often the expected value) of the certainty equivalent. Section 4.3 will further highlight some of these assumptions by examples with samples from known distributions. The confidence region estimation and hypothesis testing procedure will also be highlighted in this chapter.

4.2 Some Functional and Distributional Assumptions

4.2.1 Negative Exponential Utility

Suppose, for simplicity, that an agent has a negative exponential utility function for a random variable X, defined as

$$k(x) = \begin{cases} \delta + e^{-\theta x}, & \text{if } \theta < 0, \\ x, & \text{if } \theta = 0, \\ \delta - e^{-\theta x}, & \text{otherwise.} \end{cases} \tag{4.1}$$

Note that $\delta \geq 1$, although this parameter has no bearing on concavity measures or calculations of the certainty equivalent. This utility function exhibits the property of constant absolute risk aversion (CARA) which is represented by the parameter θ, and increasing relative risk aversion (IRRA). Thus, $\theta < 0$ represents an agent who is risk loving, $\theta = 0$ represents risk neutrality, and $\theta > 0$ represents risk aversion.

Now suppose that there are two risky alternatives, X_1 and X_2, which are normally distributed with means μ_1 and μ_2 and variances σ_1^2 and σ_2^2, respectively, and an initial wealth w_0. In this case,

$$\begin{aligned} CE_j(\theta) &= k^{-1}\{E[k(x + w_0)]\} - w_0 \\ &= \mu_j - \theta\sigma_j^2/2 \end{aligned} \qquad (4.2)$$

for $j = 1, 2$. Therefore, the agent will prefer Alternative 1 if

$$\mu_1 > \mu_2 + \frac{\theta}{2}(\sigma_1^2 - \sigma_2^2).$$

Assuming $\sigma_1 = \sigma_2$ or $\theta = 0$, the alternative with the highest expected value is preferred. If the expected values are equal, the alternative with the smallest variance is preferred for positive θ, and the alternative with the highest variance is preferred for negative θ.

In this case, if both alternatives are normally distributed with unknown parameters, method of moments procedures could be used to estimate the unknown parameters for the certainty equivalents because of their explicit form. This estimate of the certainty equivalent would be unbiased. The variance of the sample certainty equivalent could then be estimated based on unbiased moment plug-in estimators due to the independence of the sufficient statistics. If we let

$$\widehat{CE}_j(\theta) = \bar{X}_j - \frac{\theta}{2}s_j^2, \qquad (4.3)$$

the variance is then

$$Var\left[\widehat{CE}_j(\theta)\right] = \frac{\sigma_j^2}{n} + \frac{\theta^2\sigma_j^4}{2(n-1)}, \qquad (4.4)$$

which is a parabolic function of $|\theta|$. However, despite explicit forms for the mean and variance, the underlying distribution does not possess an analytically explicit expression. Additionally, it would be a nontrivial matter to estimate the variance of the sample certainty equivalent statistic,

$$\widehat{CE_j}(\theta) = u^{-1}\left\{n^{-1}\sum_{i}^{n} u(x_{ij} + w_0|\theta)\right\} - w_0, \tag{4.5}$$

from the general formula for this specification of the certainty equivalent even in the case that the distribution of the variable was normal.

To illustrate the last point, note that the form of the sample certainty equivalent given a negative exponential utility function is

$$\widehat{CE}(\theta) = \begin{cases} -\frac{1}{\theta} ln\left(\frac{1}{n}\sum_{i=1}^{n} e^{-\theta x_i}\right), & \text{if } \theta \neq 0, \\ \bar{x}, & \text{otherwise.} \end{cases} \tag{4.6}$$

If $X_1, \ldots, X_n$ are independent and identically distributed normal random variables with mean μ and variance σ^2, then $y_i = e^{-\theta x_i}$ has a lognormal distribution with mean $e^{-\theta\mu+\theta^2\sigma^2/2}$ and variance $e^{2(-\theta\mu+\theta^2\sigma^2)} - e^{-2\theta\mu+\theta^2\sigma^2}$. The sum of lognormal random variables does not have a distribution with a tractable analytical form (Milevsky and Posner 1998). Thus the variance of the logarithm of $\frac{1}{n}\sum_{i=1}^{n} y_i$ must be approximated even in the case of normality of the original data.

More generally, for the negative exponential utility function and regardless of initial wealth, the certainty equivalent is

$$CE(\theta) = \begin{cases} -\frac{1}{\theta} ln\left[M_X(-\theta)\right], & \text{if } \theta \neq 0, \\ \mu, & \text{otherwise,} \end{cases} \tag{4.7}$$

where $M_X(-\theta)$ is the moment generating function of X, if the function exists. Transformations are simple because the function is monotonically increasing for all θ. These aspects make the assumption of negative exponential utility appealing, especially in the case of normality, where the certainty equivalent is a simple function of the mean, variance, and utility parameters (cf. Hammond 1974).

These assumptions also put certainty equivalence analysis in a class of preference ranking comparable to mean-variance methods. At the same time, the complexity of the variance estimation procedure for the sample certainty equivalent creates a circumstance where the sample variability is often ignored for analytical purposes. The result in many cases is the development of efficient sets which include or exclude alternatives based on the estimates of the first sample moment of the certainty equivalent.

4.2.2 Power Utility

Another common utility function used for analyses is the power utility,

$$k(x) = \begin{cases} \frac{1}{1-\theta} x^{1-\theta}, & \text{if } \theta \neq 1, \\ \ln x, & \text{otherwise.} \end{cases} \tag{4.8}$$

This utility function exhibits constant relative risk aversion (CRRA), again represented by the parameter θ, and, for $\theta > 0$, decreasing absolute risk aversion (DARA). Unlike the negative exponential utility, this function is sensitive to the specification of initial wealth. The sample certainty equivalent for this utility function is

$$\widehat{CE}(\theta) = \begin{cases} \left[\frac{1}{n} \sum_{i=1}^{n} (x_i + w_0)^{1-\theta}\right]^{1/(1-\theta)} - w_0, & \text{if } \theta \neq 1, \\ \prod_{i=1}^{n} (x_i + w_0)^{1/n} - w_0, & \text{otherwise,} \end{cases} \tag{4.9}$$

where it exists. If $y_i = x_i + w_0$ is a random variable with expected value of μ_y and variance σ_y^2, then, using Jensen's Inequality, the certainty equivalent is bounded above by μ_y if $\theta > 0$ and below by μ_y if $\theta < 0$ for all $y_i > 0$. Again, the variance for the sample certainty equivalent may not have an explicit analytical form under many distributional assumptions.

4.2.3 Expo-Power Utility

A third example of an analytical utility function is the expo-power, presented by Saha (1993). The form of the expo-power utility is

$$k(x) = \delta - \exp(-\theta_2 x^{\theta_1}), \tag{4.10}$$

where $\delta > 1, \theta_1 \neq 0, \theta_2 \neq 0$, and $\theta_1\theta_2 > 0$. Similar to the negative exponential, the δ parameter does not impact formulations of risk indices. This function can exhibit decreasing absolute risk aversion (DARA) if $\theta_1 < 1$, CARA (or the negative exponential utility) if $\theta_1 = 1$, or increasing absolute risk aversion (IARA) if $\theta_1 > 1$. Additionally, the function models decreasing relative risk aversion (DRRA) if $\theta_2 < 0$ or increasing relative risk aversion (IRRA) if $\theta_2 > 0$. The function is quasi-concave for all positive realizations of X. The sample form of the certainty equivalent is

$$\widehat{CE}(\underline{\theta}) = \begin{cases} \left\{-\frac{1}{\theta_2}\ln\left(\frac{1}{n}\sum_{i=1}^{n}\exp\left[-\theta_2(x_i + w_0)^{\theta_1}\right]\right)\right\}^{1/\theta_1} - w_0, & \text{if } \theta_1, \theta_2 \neq 0, \\ \bar{x}, & \text{otherwise,} \end{cases} \tag{4.11}$$

where it is defined. This utility function allows for a great deal of flexibility in the assumption of an agent's preferences. It also further complicates the analytical development of the variance of the sample certainty equivalent.

These examples are by no means exhaustive, since there is an unlimited number of possible adherent functional forms for utility. These particular forms have been presented because of their extensive use in expected utility analyses. More flexible, and thus less parsimonious, utility functions are applicable in certain situations, but as the dimensionality of the parameter space increases, the interpretability of individual parameters and their respective relevant spaces, especially related to indices of risk aversion, lessens. This is typically a concern when some sort of inference is to be made about the parameter space of the utility function.

4.3 Examples of Functional and Distributional Assumptions

As an example of some of the aforementioned assumptions, let X_1 and X_2 be two independent, normally distributed alternative returns with means $\mu_1 = \mu_2 = 100$ and variances $\sigma_1^2 = 30^2$ and $\sigma_2^2 = 25^2$, respectively. The cumulative distribution functions are depicted in Figure 8. Neither alternative exhibits first degree dominance. Under the general assumption of risk aversion, X_2 SSD X_1, due solely to the lower variation of Alternative 2.

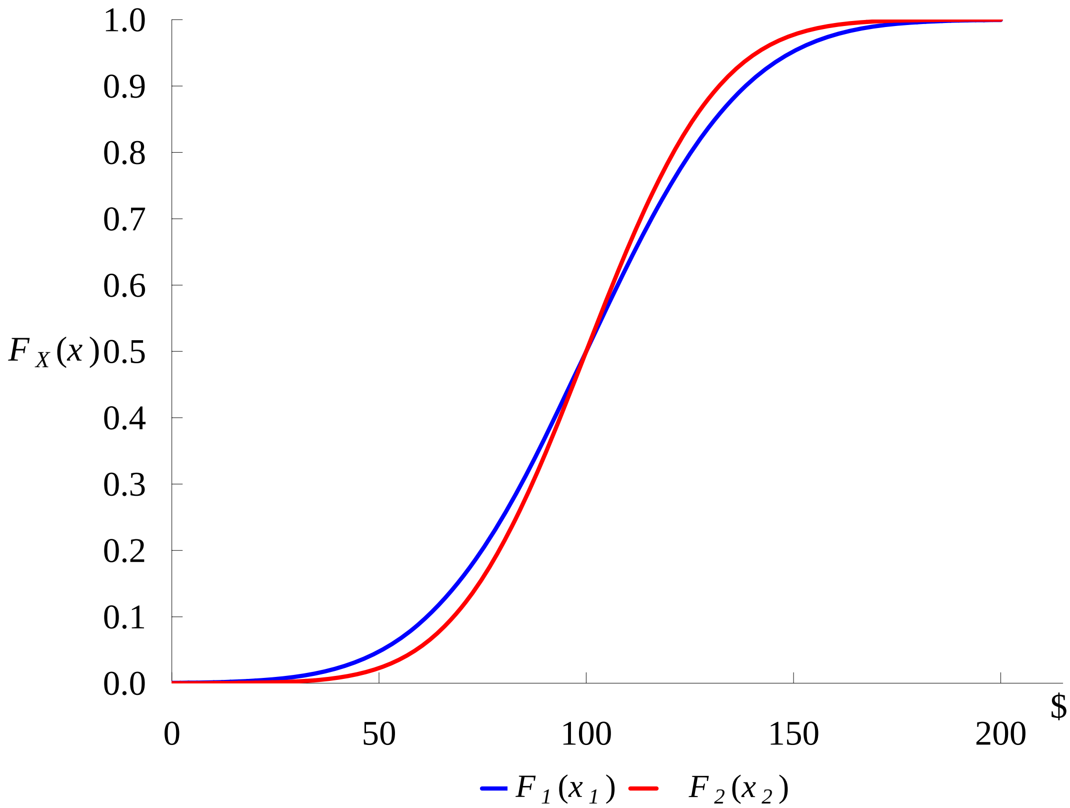


Figure 8. Cumulative distribution functions (cdfs) for the returns of $J = 2$ normally distributed alternative investments with equal means and unequal variances.

For the remainder of the chapter, samples of size $n = m = 30$ from the two distributions will be considered for empirical calculations. The base sample, used for illustrative purposes, is given in Table 6 in Appendix B. A normal probability plot of the two samples is shown in Figure 35, also in Appendix B, which illustrates sample deviations from the true distributions. The normal distributions were chosen for an example because there is a preference for one or the other over ranges of risk aversion indices that are not equal to zero based solely on the specific variability. The particular parameterizations and sample sizes were chosen because the small differences in relative variability can result in large variabilities in sample comparisons, from correct preference ordering, to ambiguity, to incorrect assessments. An initial wealth level of $w_0 = \$100$ will be assumed for the remaining analyses.

4.3.1 Negative Exponential Utility

Under the more general method of SSD(k_θ), if the agent's decision-making is modeled with the negative exponential utility function, then Alternative 1 will be preferred if $0 > \frac{275}{2}\theta$, or simply if $\theta < 0$. Therefore, risk loving agents prefer Alternative 1, risk neutral agents are indifferent between the two, and risk averse agents prefer Alternative 2, since the certainty equivalent is just a linear function of θ.

The variances of the certainty equivalents can be estimated based on the explanations in the previous section. Using the plug-in estimates for the sample certainty equivalent given normal distributions (Equation 4.3), the sample certainty equivalent is a linear combination of a normal and a χ^2 random variable. Figure 9 illustrates the certainty equivalent lines for the two series over a range of θ.

Because of the complexity of the underlying distributions of the certainty equivalents, a Monte Carlo simulation[1] was conducted to estimate the densities. Two independent samples of size $n = 30$ were used to calculate the certainty equivalents. This

[1] Simulations were conducted using Latin Hypercube random number generation. See Richardson, et. al. (2005) for procedures and documentation.

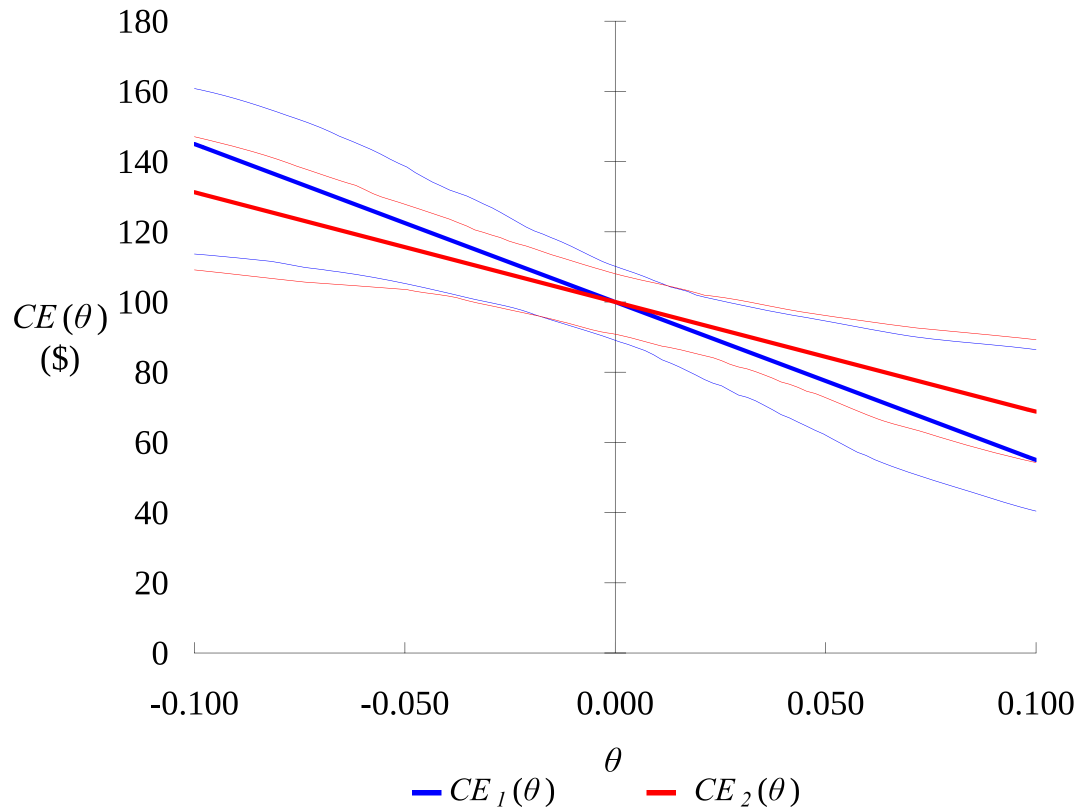
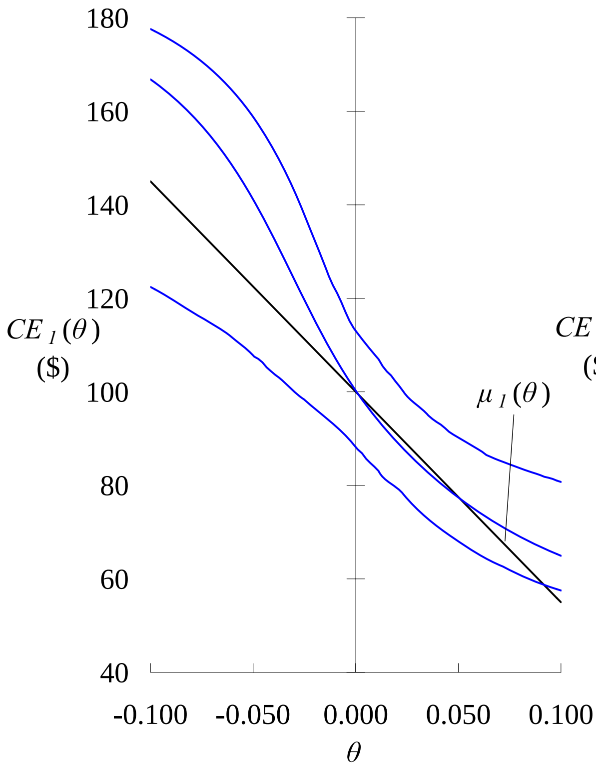


Figure 9. Stochastic efficiency with respect to a function (SERF) chart of certainty equivalents for the returns of $J = 2$ normally distributed alternative investments with approximate 95% probability mass regions. θ, an index of absolute risk aversion, is the parameter of the negative exponential utility function used in formulating the certainty equivalents.

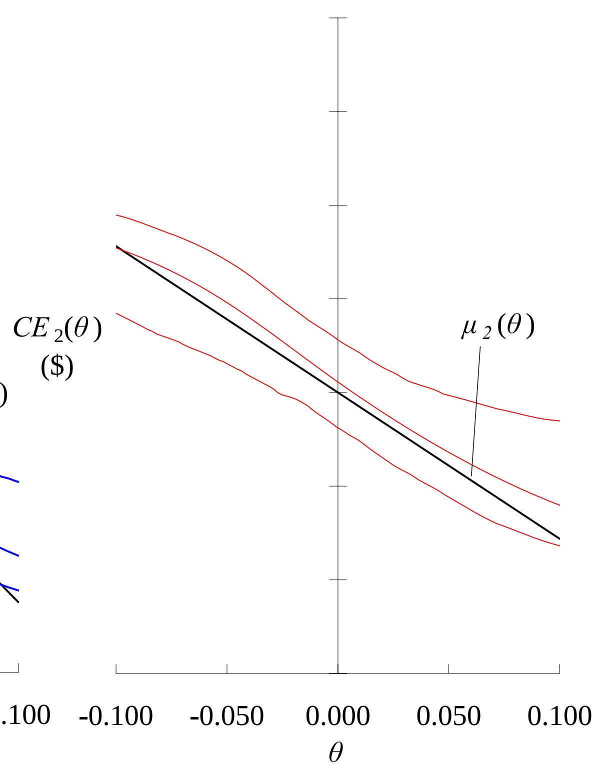
was replicated 500 times, and the associated quantiles from this simulation were used to approximate a 95% confidence region for the certainty equivalent lines. Based on the properties of the underlying distribution, the distributions become more skewed and the confidence regions widen as $|\theta|$ increases. This is also illustrated in Figure 9.

Now assuming the underlying distributions of the sample data are unknown, the resampling procedure will be used for estimation of the confidence regions. For the bootstrap method, the sample estimate of the certainty equivalent was calculated for

500 bootstrap samples of size $n_B \equiv 30$. As shown in Figure 10, the sample mean and the variance estimates for the certainty equivalents are highly influenced by the particular sample. The sample estimates are closer to the true value for θ near zero, but as $|\theta|$ increases, the sample mean values move away from the true values for this particular sample.



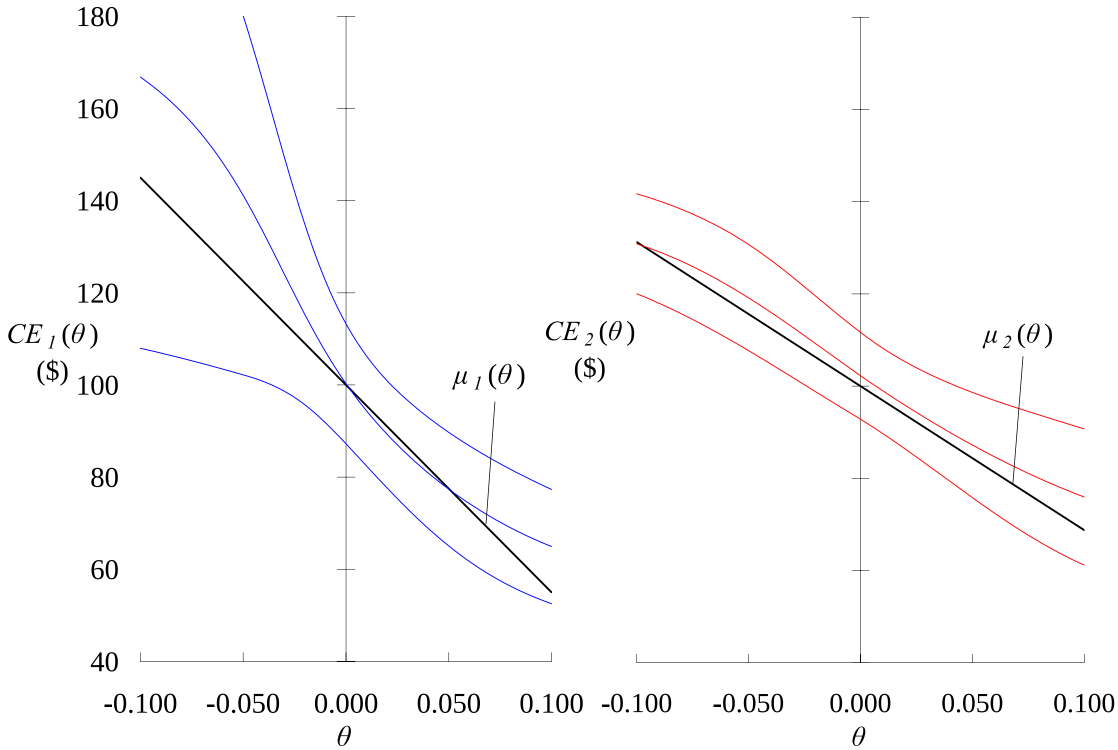
(a) SERF Chart for Alternative 1



(b) SERF Chart for Alternative 2

Figure 10. *Stochastic efficiency with respect to a function (SERF) charts of the expected certainty equivalents ($CE_1(\theta)$ and $CE_2(\theta)$), the mean sample certainty equivalents, and 95% bootstrap confidence regions based on a sample of size $n = 30$ for the returns of two normally distributed alternative investments (X_1 and X_2) under a negative exponential utility function given parameter θ.*

Paying particular attention to Figure 10(a), the true certainty equivalent value extends beyond the coverage of the bootstrap confidence region for $\theta > 0.100$. Since the true expected value of the certainty equivalent is in the range of $(-\infty, \infty)$ as a function of θ for data from a normal distribution, the bootstrap intervals, especially in small samples, tend to perform poorly for large $|\theta|$. Comparing Figure 9 to Figure 10, the bootstrap confidence regions do follow the general shape of the simulated confidence regions in terms of skewness and width.



(a) SERF Chart for Alternative 1 (b) SERF Chart for Alternative 2

Figure 11. *Stochastic efficiency with respect to a function (SERF) charts of the expected certainty equivalents ($CE_1(\theta)$ and $CE_2(\theta)$), the mean sample certainty equivalents, and approximate 95% confidence regions based on a sample of size $n = 30$ and jackknife variance estimates for the returns of two normally distributed alternative investments (X_1 and X_2) under a negative exponential utility function given parameter θ.*

As a second example, using the same sample data, the jackknife procedure will be used to estimate the variance of the sample certainty equivalent. Figure 11 illustrates the asymptotic 95% confidence regions produced as a result of using normal quantiles and assuming the sample mean and jackknife variance of the certainty equivalent represent the true values, respectively. Comparing Figure 11 to Figure 10, the coverage area for the jackknife confidence region includes the true value over the range of θ represented in both cases, while the bootstrap does not. The region based on the jackknife method increases as a function of $|\theta|$ just as did the simulated distribution, but the jackknife procedure is not restricted by the sample as is the case with the bootstrap confidence region.

Based on the sample data, the hypothesis of preference with respect to negative exponential utility will now be examined. Since the sample size of each of the two alternatives is $n = 30$, the statistic presented in Chapter III, $T_N(\theta)$, is a function of $N = 2n = 60$ observations, and will be denoted $T_{60}(\theta)$. Using the jackknife estimation procedure for the sample variance, the value of the test statistic at varying θ values is presented in Figure 12.

Recall that $\mu_1 = \mu_2 = 100$ and $\sigma_1^2 = 30^2$ and $\sigma_2^2 = 25^2$. So, true preferences, in terms of stochastic efficiency with respect to a function, are that X_1 is preferred to X_2 for $\theta \in (-\infty, 0)$, and X_2 is preferred to X_1 for $\theta \in (0, \infty)$. Although the statistic is less than zero for $\theta > 0$, at a 95% confidence level, the null hypothesis of indifference between the two alternatives is not rejected in favor of $H_1 : X_2 \ SSD(k_\theta) \ X_1$ for the particular sample.

Using Monte Carlo simulation, multiple samples of sizes $n = 30$ were drawn and the test statistic was calculated across a range of θ values for each of 500 samples. The results are seen in Figure 13. The replications were used to approximate the distribution of $T_{60}(\theta)$. Under the null hypothesis of indifference between the two alternatives, the mean value of the test statistic should be zero for any given θ. The center of the distribution shifts toward the rejection region for a sub-set of θ where $X_2 \ SSD(k_\theta) \ X_1$. The true difference in preference between the two is based solely on coefficients of variation of 30% and 25% for X_1 and X_2, respectively.

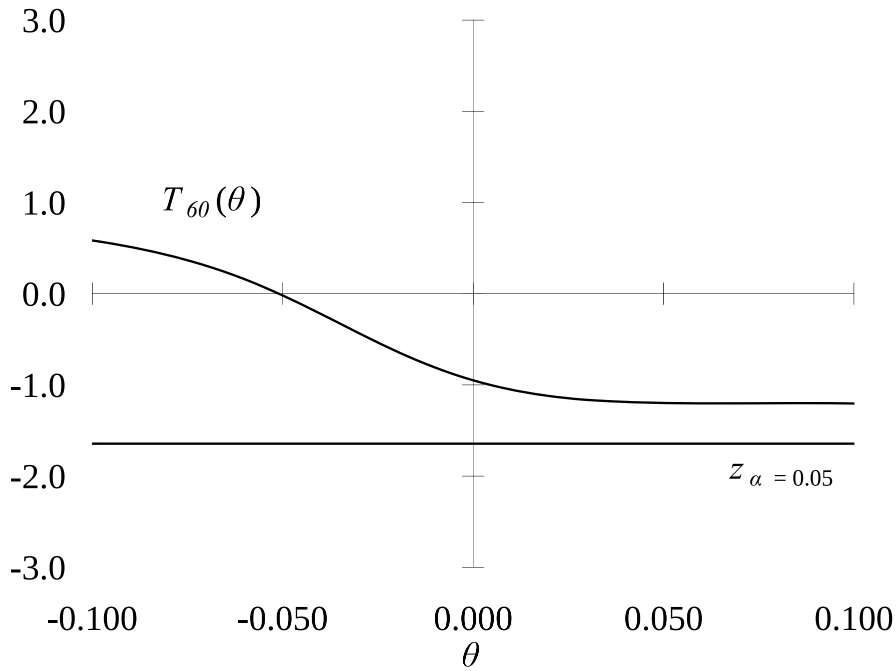


Figure 12. *Test statistic, $T_{60}(\theta)$, based on a single sample of $n = 30$ returns of two normally distributed alternative investments, X_1 and X_2. The null hypothesis of indifference between the two alternatives is rejected in favor of $H_1 : X_2\ SSD(k_\theta)\ X_1$ in large samples for values of $T_{60}(\theta) < z_{\alpha=0.05}$, where k_θ is the negative exponential utility function with θ as the coefficient of absolute risk aversion.*

For a basis of comparison, another simulation of the same type was conducted, but in this case for the two normal series, let $\mu_1 = \mu_2 = 100$ and $\sigma_1^2 = \sigma_2^2 = 30^2$. A plot of the simulated mean and probability interval of the test statistic based on the two identical parameter samples is given in Figure 14. As might be expected, regardless of the value of θ presented, $T_{60}(\theta)$ appears to have an approximate standard normal distribution. In general, and even under the null hypothesis, the distribution can degenerate for some parameterizations of θ due to the extreme curvature of the utility function.

Along the lines of comparisons using known distributions, the mean empirical power functions of the hypothesis given varying sample sizes as a function of σ, the

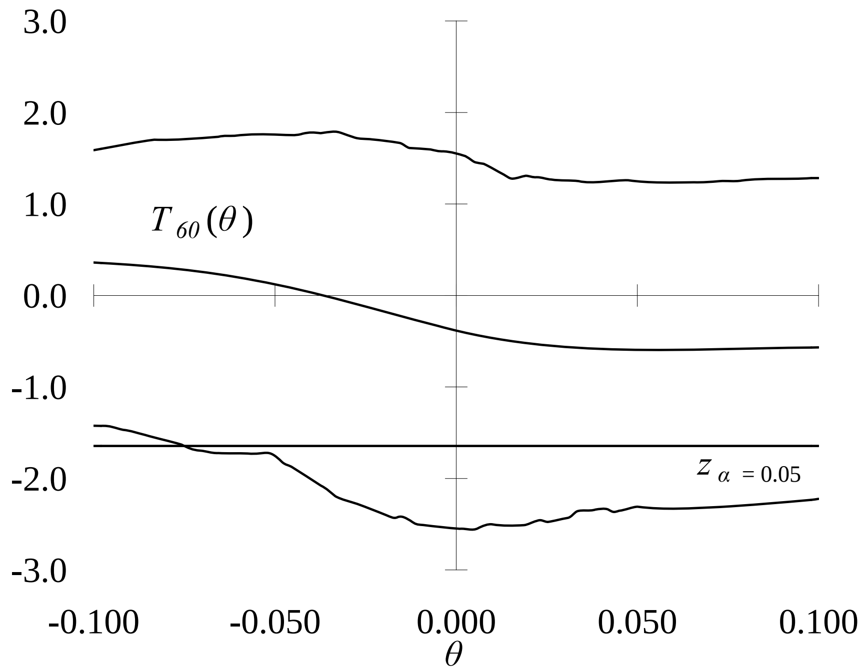


Figure 13. *Mean and 95% probability mass of the test statistic, $T_{60}(\theta)$, approximated from a Monte Carlo simulation of 500 iterations of samples of $n = 30$ returns of two normally distributed alternative investments, X_1 and X_2. The null hypothesis of indifference between the two alternatives is rejected in favor of $H_1 : X_2\ SSD(k_\theta)\ X_1$ in large samples for values of $T_{60}(\theta) < z_{\alpha=0.05}$, where k_θ is the negative exponential utility function with θ as the coefficient of absolute risk aversion.*

standard deviation of alternatives, is given in Figure 15. Recall that for normal distributions with equal means and assuming a negative exponential utility function with a fixed value of θ, alternatives are ranked solely on the variability. For $\theta < 0$, more variability is preferred, and for $\theta > 0$, less variability is desired. Given two normally distributed alternatives with equal means, the hypothesis tests from which Figure 15 is derived are expressed as $X_j\ SSD(k_\theta)\ X_l$ under the negative exponential utility function assuming risk averse agents with $\theta = 0.005$. X_l has variance $\sigma_l^2 = 30^2$. Each of the j alternatives to be tested have variances σ_j^2. The power of the test is shown as a function of the true standard deviation parameter. The power increases

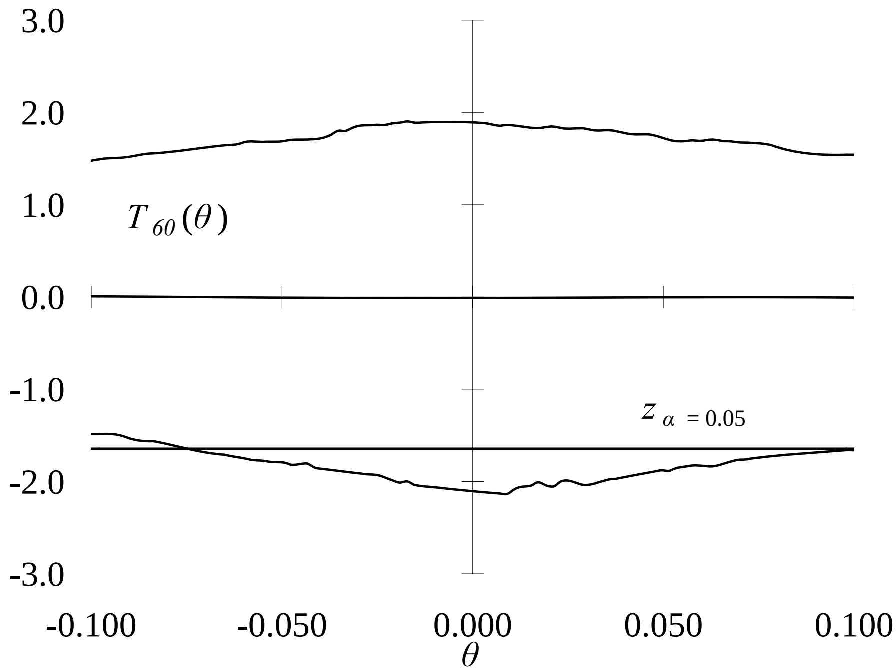


Figure 14. Mean and 95% probability mass of the test statistic, $T_{60}(\theta)$, approximated from a Monte Carlo simulation of 500 iterations of samples of $n = 30$ returns of two normally distributed alternative investments, X_1 and X_2, with equal means and variances. The null hypothesis of indifference between the two alternatives is rejected in favor of $H_1 : X_2\ SSD(k_\theta)\ X_1$ in large samples for values of $T_{60}(\theta) < z_{\alpha=0.05}$, where k_θ is the negative exponential utility function with θ as the coefficient of absolute risk aversion.

as the variance decreases, and the null hypothesis of indifference is more likely to be rejected. Similarly, the relative power is higher for increased sample sizes. To compare further, a sample size of $n = 100$ was used to calculate the statistic $T_{200}(\theta)$ for a range of θ under the original parametric specifications of the two distributions. Another 500 iteration simulation was conducted to estimate the distribution of the test statistic. The results are shown in Figure 16. The center of the distribution is more clearly a function of θ relative to that presented in Figure 13. The null hypothesis is rejected more frequently over the same range of θ in the larger sample simulation.

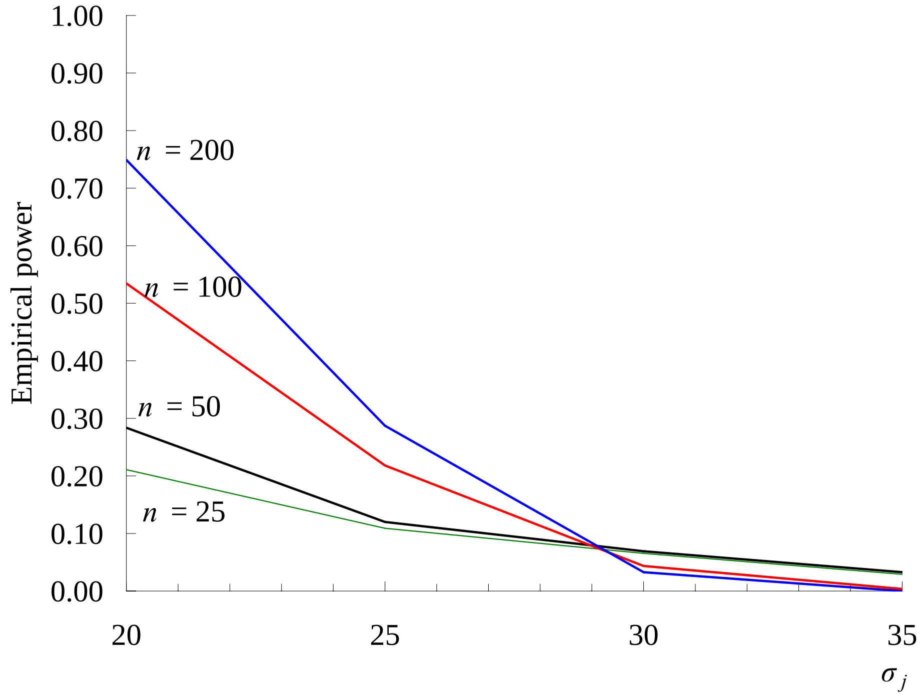


Figure 15. Empirical power function based on the means of Monte Carlo simulations of sample sizes $n = 25, 50, 100,$ and 200. *The hypothesis,* $H_1 : X_j\ SSD(k_\theta)\ X_l$, *at a 95% confidence level is rejected in large samples when* $T_N < z_{\alpha=0.05}$. $X_l \sim N(100, 30^2)$ *and* $X_j \sim N(100, \sigma_j^2)$. *The empirical power is expressed as a function of* σ_j. *The negative exponential utility function is shown for* $\theta = 0.005$.

It is interesting to note that, as can be seen in Figure 16, the mean of the test statistic approaches zero from below as θ increases. This is due to the fact that, for a given fixed (finite) sample from a random variable X, as $\theta \to \infty$, $E[k(X|\theta)] \to \delta$ (recalling that δ is the intercept parameter in the negative exponential utility formula, usually taken to be one) and $Var[k(X|\theta)] \to 0$. If an alternative dominates another alternative with respect to this class of utility functions, $T_N(\theta)$ will be marginally negative and approach zero as θ increases due to the extreme concavity imposed.

On the other extreme, the test statistic approaches infinity as $\theta \to -\infty$. Recall that for this utility function, θ represents absolute risk aversion, so for $\theta \to \pm\infty$, the agent's risk preference is such that the only values of interest from the distribution

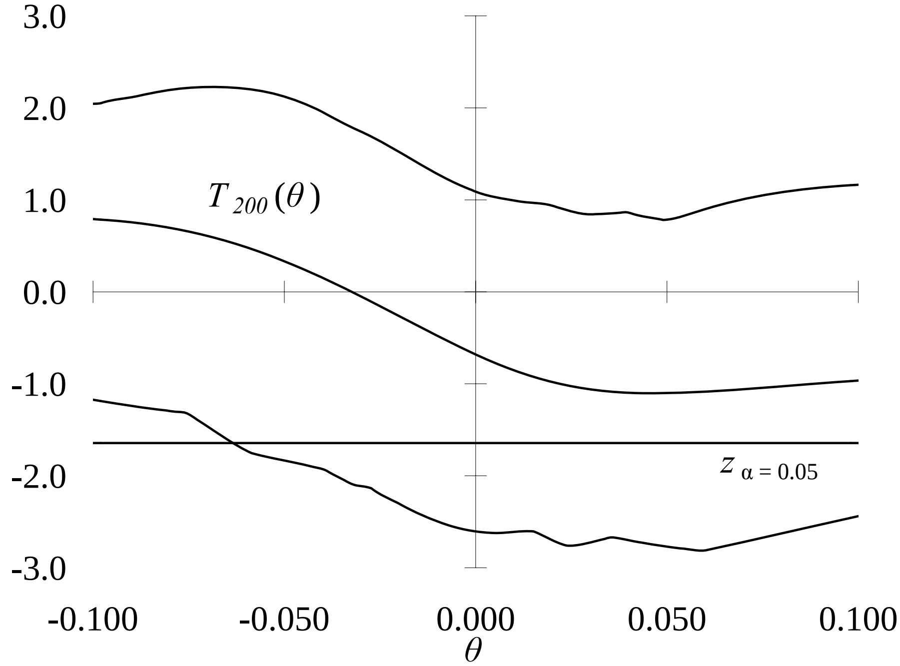


Figure 16. Mean and 95% probability mass of the test statistic, $T_{60}(\theta)$, approximated from a Monte Carlo simulation of 500 iterations of samples of $n = 100$ returns of two normally distributed alternative investments, X_1 and X_2. The null hypothesis of indifference between the two alternatives is rejected in favor of $H_1 : X_2\ SSD(k_\theta)\ X_1$ in large samples for values of $T_{60}(\theta) < z_{\alpha=0.05}$, where k_θ is the negative exponential utility function with θ as the coefficient of absolute risk aversion.

of a risky prospect are one of the two extreme values, depending on the direction of approach of θ. For a fixed sample, the preference weight shifts soley to the maximum value as $\theta \to -\infty$. Under the hypothesis $H_1 : X\ SSD(k_\theta)\ Y$, if the maximum value of the two samples belongs to X (and the maxima are not equal), then $T_N(\theta) \to \infty$ as $\theta \to -\infty$. Otherwise, $T_N(\theta) \to -\infty$ as $\theta \to -\infty$.

In light of this, there may only be a finite range of θ where the hypothesis of indifference can be rejected for utility functions that are only strictly concave or convex depending on the value of θ. This is, in one respect, a means to infer a possible relevant range of θ, and thus the risk indices as well, if preferences between two alternatives do actually exist.

4.3.2 Power Utility

Using a power utility function to model preferences, the results are similar to those in the use of the negative exponential. Assuming an initial wealth of \$100, Alternative 1 is preferred to Alternative 2 over $\theta < 0$ and the reverse is true for all $\theta > 0$, where in this case θ is the relative risk aversion coefficient. Although the results are similar to those assuming the negative exponential utility, the magnitude of θ is different for the power utility.

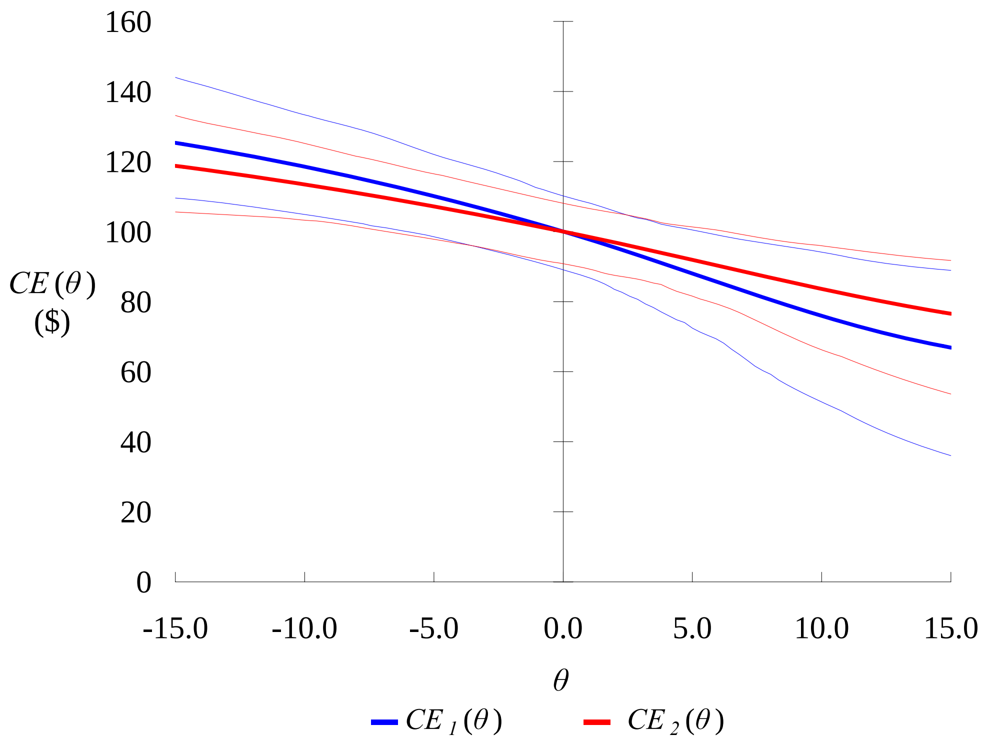


Figure 17. Stochastic efficiency with respect to a function (SERF) chart of certainty equivalents for the returns of $J = 2$ normally distributed alternative investments with approximate 95% probability mass regions. θ, an index of relative risk aversion, is the parameter of the power utility function used in formulating the certainty equivalents.

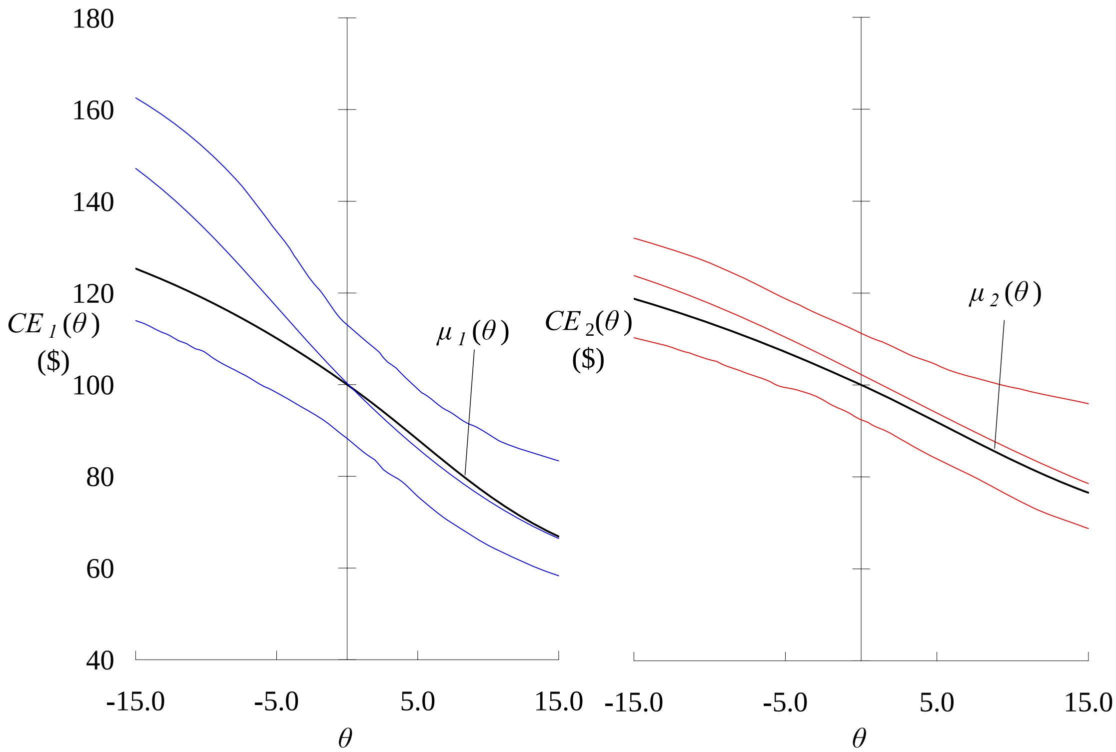
A similar Monte Carlo sampling procedure to the ones implemented previously was used to estimate the mean certainty equivalent line over a sub-set of θ as well as an approximate 95% probability mass formed from the sample quantiles. The results are shown in Figure 17. The relevant range for θ in this case is presented as $(-15, 15)$.

From Figure 17 it can be seen that, with normality and the power utility function, the distribution of the certainty equivalent appears to be more symmetric than that when the negative exponential utility function is assumed. Since the variation of X_2 is smaller than that of X_1, the corresponding variability of the certainty equivalent is lower than that of the certainty equivalent of X_1 for all θ. To make the results comparable to those when assuming the negative exponential utility, note that $r_A = r_R/E(X_1) = \theta/100$.

Again, assuming the underlying distribution is unknown, the certainty equivalent and corresponding confidence regions were calculated. Figure 18 presents the confidence regions based on bootstrap simulations of the sample data of size $n = 30$. Figure 19 shows the confidence regions based on jackknife estimates of the variance of the sample certainty equivalent under the assumption of asymptotic normality.

In this case of normal distributions and a power utility function, the confidence regions using the bootstrap and jackknife procedures provide more similar coverage areas than the methods applied under the negative exponential utility assumption for this particular sample. The drawbacks in each method are still present: the bootstrap confidence regions are bound by the sample extreme values and the jackknife regions assume symmetric distributions and have the potential to grow very large as a function of $|\theta|$.

Turning again to testing procedures, the hypothesis that Alternative 2 stochastically dominates Alternative 1 with respect to the power utility function given θ will be tested. The null hypothesis of indifference between the two alternatives is not re-

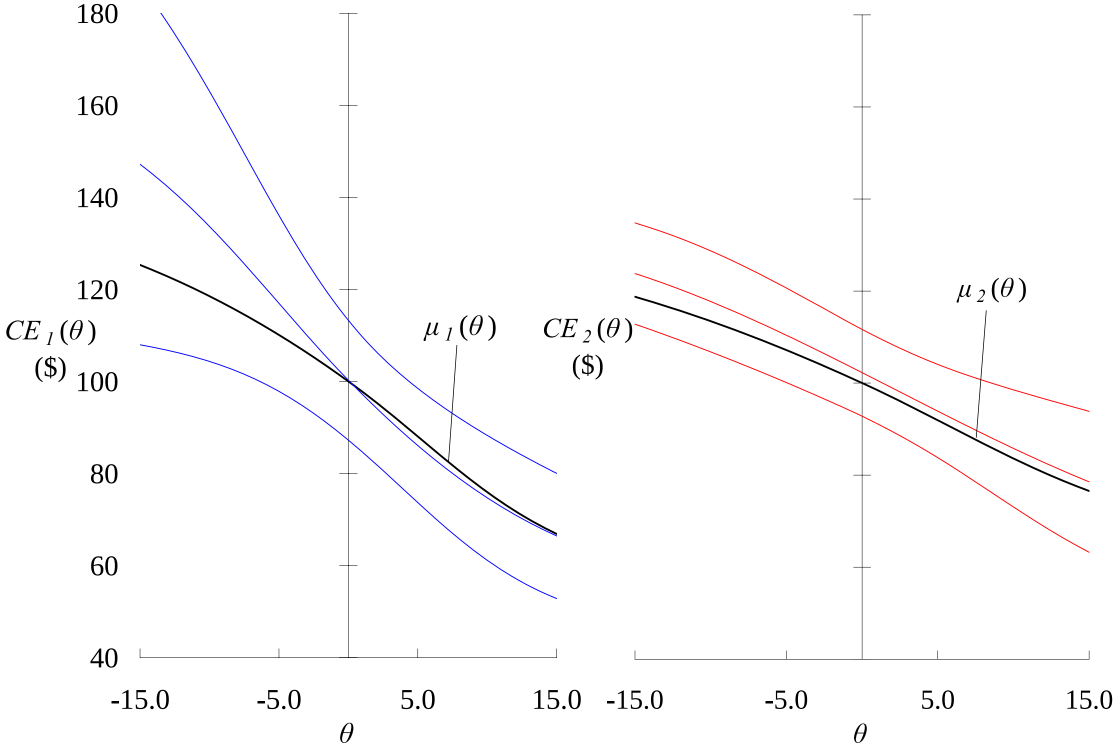


(a) SERF Chart for Alternative 1 (b) SERF Chart for Alternative 2

Figure 18. *Stochastic efficiency with respect to a function (SERF) charts of the expected certainty equivalents ($CE_1(\theta)$ and $CE_2(\theta)$), the mean sample certainty equivalents, and 95% bootstrap confidence regions based on a sample of size $n = 30$ for the returns of two normally distributed alternative investments (X_1 and X_2) under a power utility function given parameter θ.*

jected for the range $\theta \in (-15, 15)$ at a 95% asymptotic confidence level. The results of the test for the single sample are presented in Figure 20. This is the same result as that under the negative exponential utility assumption. Replications with 500 samples of $n = 30$ of the two alternatives were used to approximate the distribution of the certainty equivalents. As seen in Figure 21, the resulting distribution is similar to that presented previously.

The mean empirical power functions for the test statistic $T_N(\theta)$ given varying sample sizes as a function of σ, the standard deviation of alternatives is illustrated



(a) SERF Chart for Alternative 1 (b) SERF Chart for Alternative 2

Figure 19. *Stochastic efficiency with respect to a function (SERF) charts of the expected certainty equivalents ($CE_1(\theta)$ and $CE_2(\theta)$), the mean sample certainty equivalents, and approximate 95% confidence regions based on a sample of size $n = 30$ and jackknife variance estimates for the returns of two normally distributed alternative investments (X_1 and X_2) under a power utility function given parameter θ.*

in Figure 22. Given two normally distributed alternatives with equal means, the hypothesis tests are expressed as $X_j\ SSD(k_\theta)\ X_l$ under the power utility function assuming $\theta = 7.5$ (or risk averse agents who prefer less risk, c.p). X_l has variance $\sigma_l^2 = 30^2$ and the variance of X_j depends on j. The power of the test is shown as a function of the true standard deviation; as the variance decreases, the power increases, and the null hypothesis of indifference is more likely to be rejected.

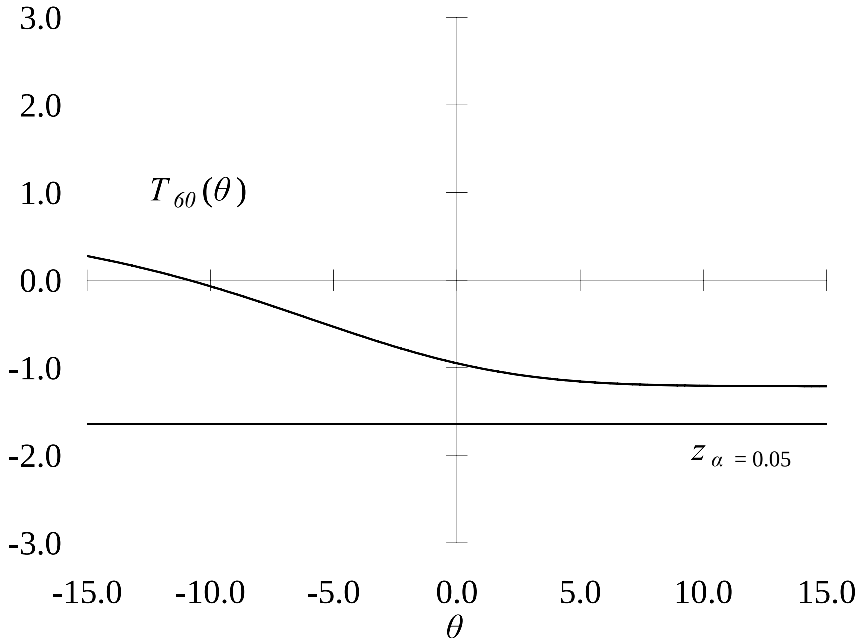


Figure 20. *Test statistic, $T_{60}(\theta)$, based on a single sample of $n = 30$ returns of two normally distributed alternative investments, X_1 and X_2, assuming a power utility function for preference modeling.*

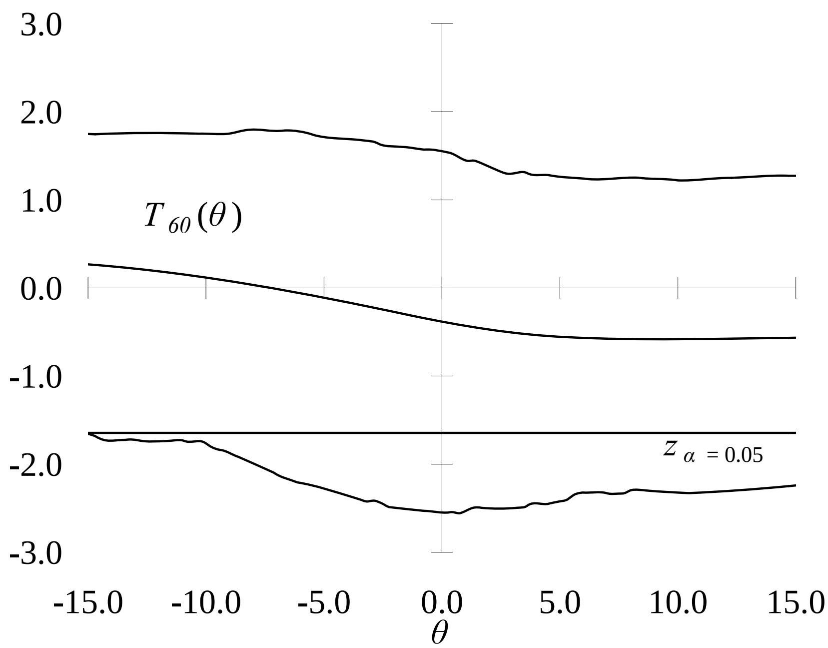


Figure 21. *Mean and 95% probability mass of the test statistic, $T_{60}(\theta)$, approximated from a Monte Carlo simulation of 500 iterations of samples of $n = 30$ returns of two normally distributed alternative investments, X_1 and X_2, assuming a power utility function for preference modeling.*

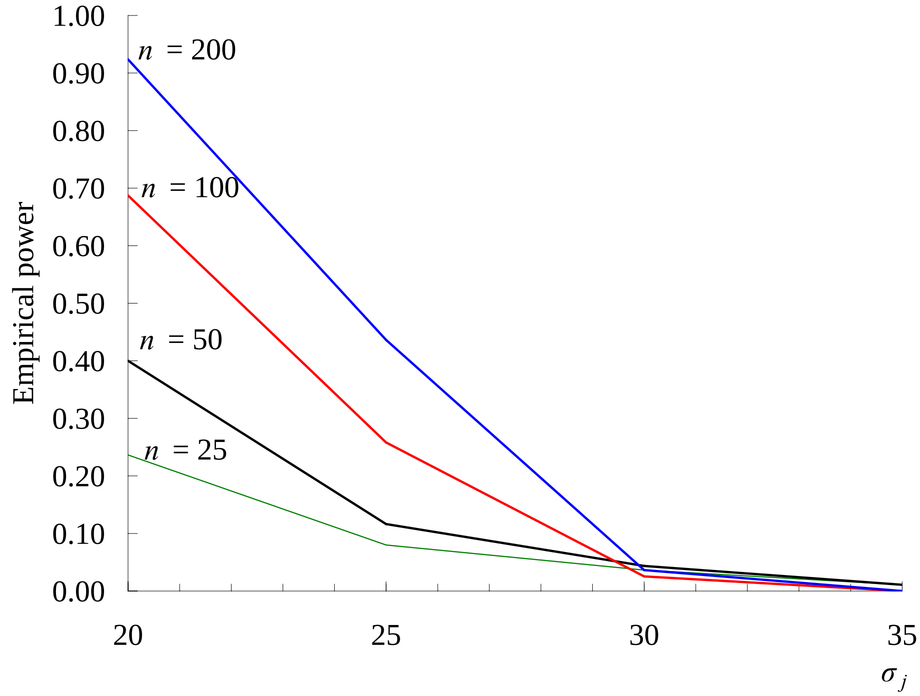
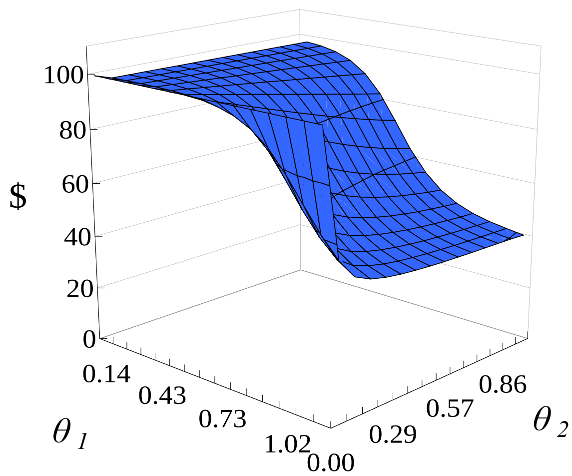


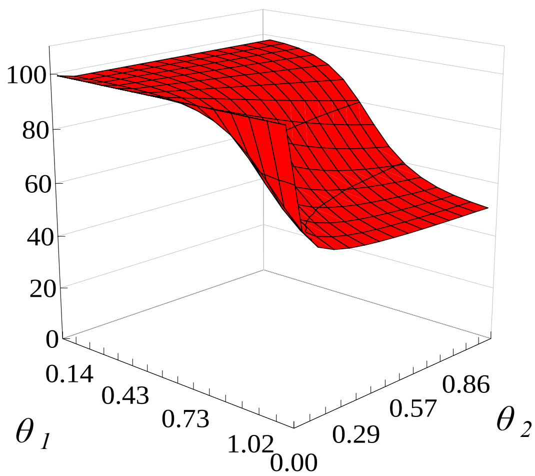
Figure 22. Empirical power function based on the means of Monte Carlo simulations of sample sizes $n = 25, 50, 100,$ and 200. The hypothesis, $H_1 : X_j\ SSD(k_\theta)\ X_l$, at a 95% confidence level is rejected in large samples when $T_N < z_{\alpha=0.05}$. $X_l \sim N(100, 30^2)$ and $X_j \sim N(100, \sigma_j^2)$. The empirical power is expressed as a function of σ_j. The power utility function is shown for $\theta = 7.5$.

4.3.3 Expo-Power Utility

Again, two independent samples from X_1 and X_2 of size $n = 30$ will be used to illustrate the assumption of an exponential utility function. A Monte Carlo simulation was used to generate 500 iterations of sample SERF surface estimates based on the $n = 30$ samples. Figure 23 illustrates the mean sample certainty equivalent surfaces over a range of the $\theta = (\theta_1, \theta_2)$ parameter space for the two samples. Under the assumption of this utility function, an agent is seen to be more risk averse as θ_1 and θ_2 increase, either simultaneously or given a fixed value for one over the prescribed parameter space.

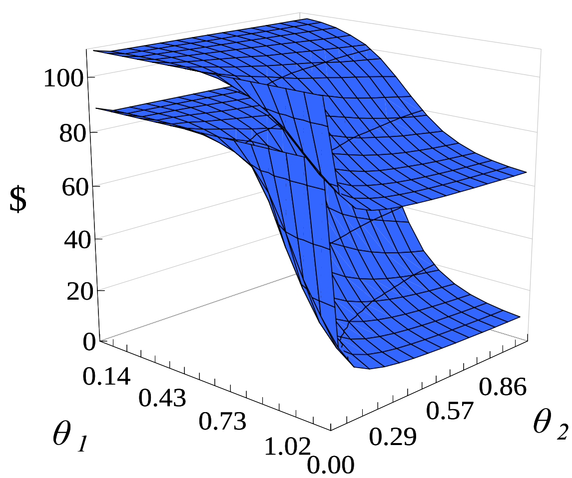


(a) SERF Chart for Alternative 1

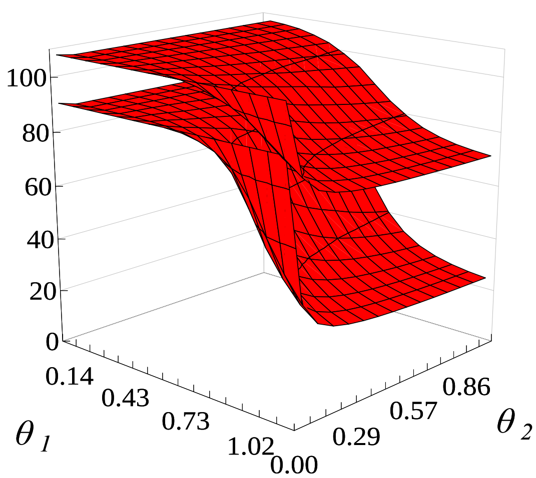


(b) SERF Chart for Alternative 2

Figure 23. Stochastic efficiency with respect to a function (SERF) chart of mean certainty equivalents for the returns of two normally distributed alternative investments (X_1 and X_2) under an expo-power utility function given parameter $\theta = (\theta_1, \theta_2)$.



(a) SERF Chart for Alternative 1



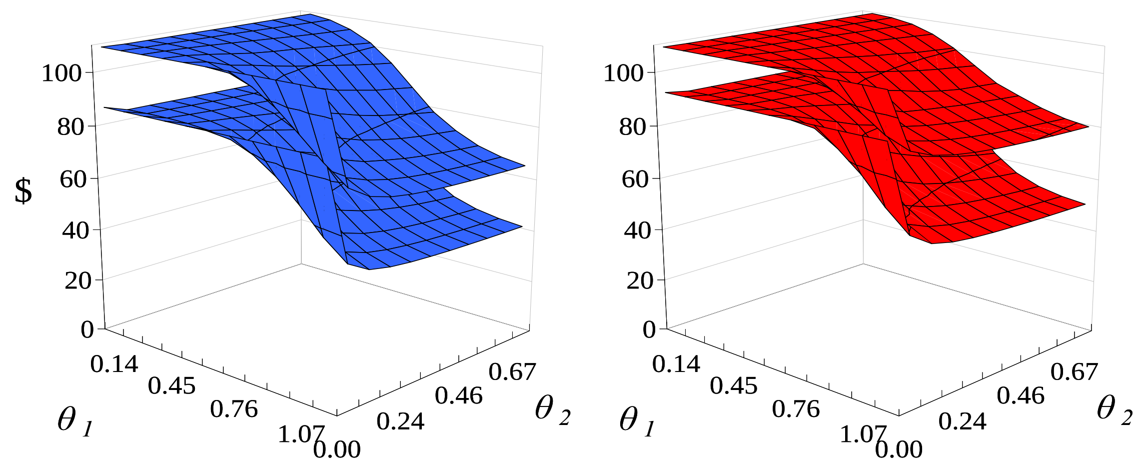
(b) SERF Chart for Alternative 2

Figure 24. Stochastic efficiency with respect to a function (SERF) chart of the approximate 95% confidence region of the certainty equivalents for the returns of two normally distributed alternative investments (X_1 and X_2) based on a sample of $n = 30$ observations of each under an expo-power utility function given parameter $\theta = (\theta_1, \theta_2)$.

Comparing the two mean certainty equivalent surfaces, there is no clearly dominant alternative, and thus the efficient set allocation depends on the subset of θ in question. Quantiles from the simulation were used to approximate a 95% confidence region for the surfaces. An example of this is illustrated in Figure 24. Note that, as in the previous example, the distribution of the surface flattens out as the parameter space moves away from the origin. This occurs in general for many monotonic utility functions, and it is important to be mindful of that when operating within certain areas of the parameter space. When dealing with parameter spaces of higher relative magnitude, the confidence levels of the surfaces make discerning between alternatives for allocation to the efficient set less clear.

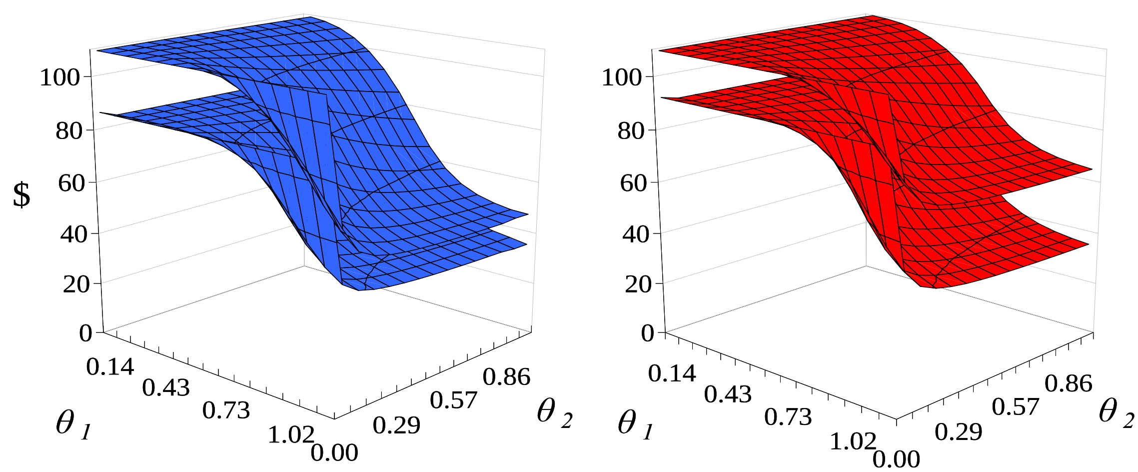
The approximate 95% confidence regions based on the bootstrap method and the jackknife variance estimation procedure were calculated for the two alternatives. Both results are based on the samples of $n = 30$ observations from two normal distributions used previously. The confidence regions using the boostrap and the jackknife methods are illustrated in Figures 25 and 26, respectively. Note that, as in previous examples, the bootstrap confidence regions plateau at the minimum observed values as θ_1 and θ_2 increase over the positive range of θ. The confidence regions based on the jackknife variance estimation procedure follow the general shape of those in Figure 25, but the region widths are generally smaller than the distributional estimates.

The jackknife variance estimation procedure, like that of the bootstrap method, is highly dependent on the specific sample. The true underlying variance of the certainty equivalent increases as θ_1 and θ_2 increase over the respective positive ranges. In Figure 26, the variance as a function of θ decreases for Alternative 1 and increases for Alternative 2 as θ_1 and θ_2 increase. As the sample data is translated through two parameter dimensions, the result is highly sensitive to the relative variability within the sample. As θ_1 and θ_2 increase, the sample certainty equivalent moves toward the



(a) SERF Chart for Alternative 1 (b) SERF Chart for Alternative 2

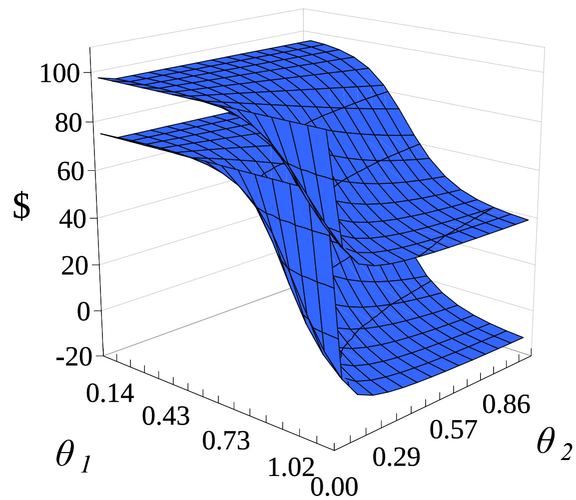
Figure 25. Stochastic efficiency with respect to a function (SERF) chart of the 95% bootstrap confidence region of the certainty equivalents for the returns of two normally distributed alternative investments (X_1 and X_2) based on a sample of $n = 30$ observations of each under an expo-power utility function given parameter $\theta = (\theta_1, \theta_2)$.



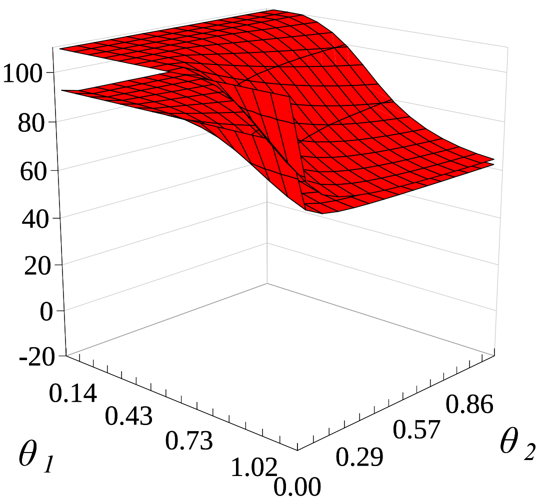
(a) SERF Chart for Alternative 1 (b) SERF Chart for Alternative 2

Figure 26. Stochastic efficiency with respect to a function (SERF) chart of the approximate 95% confidence region based on jackknife estimates of variances of the certainty equivalents for the returns of two normally distributed alternative investments (X_1 and X_2) based on a sample of $n = 30$ observations of each under an expo-power utility function given parameter $\theta = (\theta_1, \theta_2)$.

sample minimum due to effective re-weighting. Similarly, the relative within-sample variability is re-weighted based on the value of θ. Thus, the specific variability within the tails of the sample strongly influences the sample jackknife variance estimation.



(a) SERF Chart for Alternative 1



(b) SERF Chart for Alternative 2

Figure 27. Stochastic efficiency with respect to a function (SERF) chart of the approximate 95% confidence region based on jackknife estimates of variances of the certainty equivalents for the returns of two normally distributed alternative investments (X_1 and X_2) based on an alternative sample of $n = 30$ observations of each under an expo-power utility function given parameter $\theta = (\theta_1, \theta_2)$.

As a further example, a second sample was drawn to compare to the base sample. Confidence regions based on the jackknife variance estimation procedure are shown in Figure 27. Note that, based on the new sample, the change in variance as a function of θ has switched direction for each alternative. Thus, for the first sample, the regions estimated for Alternative 2 are more similar to the simulated regions in coverage, and for the second, the estimated regions for Alternative 1 are more similar to the simulated regions based on the jackknife estimation. So, at least for this two parameter utility transformation, the confidence regions based on jackknife variance estimation are, themselves, highly variable.

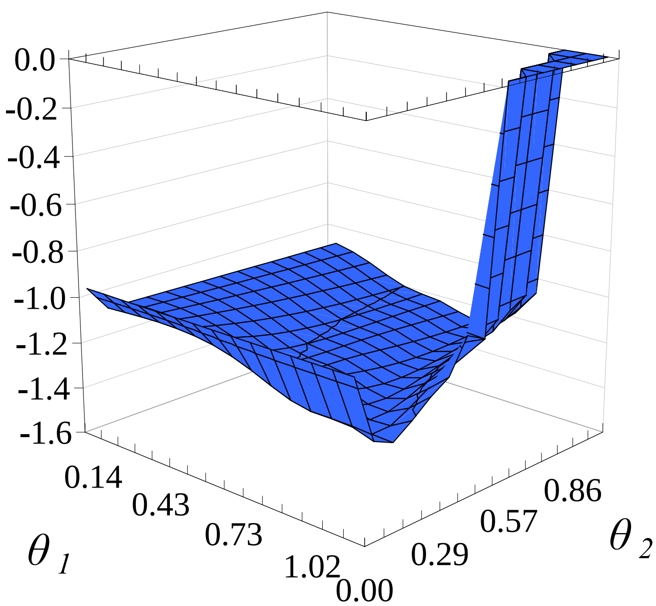
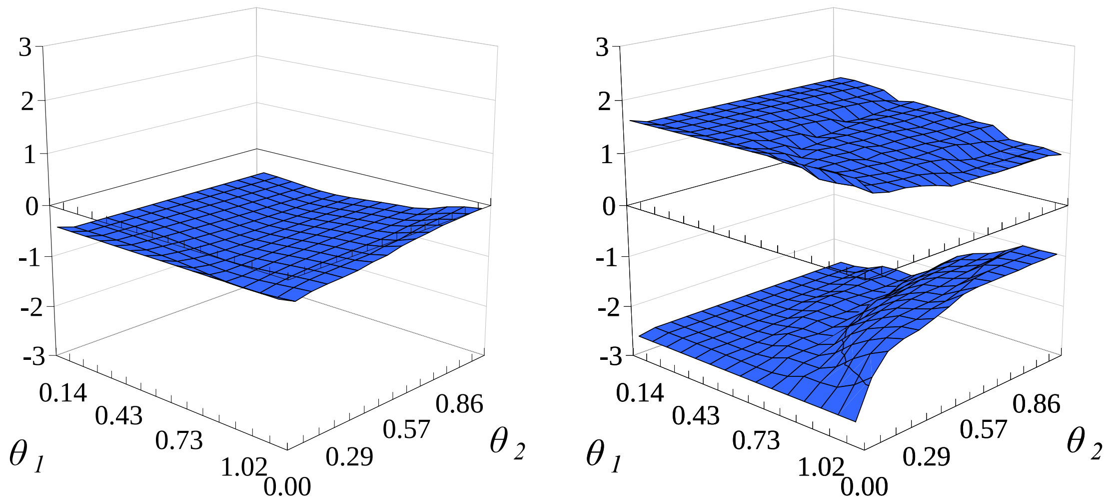


Figure 28. *Test statistic, $T_{60}(\theta)$, based on a single sample of $n = 30$ returns of two normally distributed alternative investments, X_1 and X_2. The null hypothesis of indifference between the two alternatives is rejected in favor of $H_1 : X_2\ SSD(k_\theta)\ X_1$ in large samples for values of $T_{60}(\theta) < z_{\alpha=0.05} \approx -1.645$, where k_θ is the expo-power utility function with parameters θ_1 and θ_2.*

The hypothesis that Alternative 2 is preferred to Alternative 1 with respect to expo-power utility, $H_1 : X_2\ SSD(k_\theta)\ X_1$, was tested for the two samples from normal distributions. For this hypothesis, the parameter space of interest is $\Theta_1 = \{\theta_1, \theta_2 : \theta_1 > 0, \theta_2 > 0\}$. Recall that this parameter space represents risk averse preferences, and increasing one or both of the parameters leads to an increase in risk aversion. Similar to the two previous examples, the null hypothesis of indifference is not rejected for this particular sample. The surface of the test statistic $T_{60}(\theta)$ is illustrated in Figure 28.

For a sense of the distribution of $T_{60}(\theta)$, a Monte Carlo simulation was conducted to draw 500 samples of size $n = 30$ of each of the two normally distributed alternatives. $T_{60}(\theta)$ was calculated at each iteration and the mean and approximate 95% confidence



(a) Mean of $T_{60}(\theta)$ (b) 95% Confidence Region for $T_{60}(\theta)$

Figure 29. Mean and 95% probability mass of the test statistic, $T_{60}(\theta)$, approximated from a Monte Carlo simulation of 500 iterations of samples of $n = 30$ returns of two normally distributed alternative investments, X_1 and X_2. The null hypothesis of indifference between the two alternatives is rejected in favor of $H_1 : X_2\ SSD(k_\theta)\ X_1$ in large samples for values of $T_{60}(\theta) < z_{\alpha=0.05} \approx -1.645$, where k_θ is the expo-power utility function with parameters θ_1 and θ_2.

region based on sample quantiles were estimated. This is illustrated in Figure 29. The estimated mean surface of the statistic is negative over the presented range, indicating a slight preference toward Alternative 2. The results are comparable to those of the previous two utility assumptions in the frequency of rejecting the null hypothesis and the tendency of the confidence region to shrink as the parameters increased in magnitude in the positive quadrant.

As also was true for the previous utility assumptions, as $\theta_1 \to \infty$ and $\theta_2 \to \infty$, the utility transformations for the two alternatives each approach δ, the intercept parameter of the expo-power utility functions. The variances of the utility transformations approach zero in the same limits. There is then a limited range of $\theta = (\theta_1, \theta_2)$ where the null hypothesis can be rejected before the test statistic degenerates.

4.3.4 Discussion and Conclusions

In this chapter, some common assumptions involved in the analysis of expected utility preference ranking have been described and examined. Several alternatives under varying specifications of the distributional assumption of normality were compared with respect to utility. Under these specifications, three commonly used utility assumptions in expected utility analyses have been examined. It has been demonstrated that even under the assumption of normality, which is completely specified by the first two moments, explicit calculations of preference ranking methods based on certainty equivalents are not easily derived.

Even more complicated is the estimation of the variance of the certainty equivalents. Resampling procedures, specifically bootstrap resampling to estimate the mean and variance of the certainty equivalent and jackknife estimations of the variance, have been demonstrated to provide adequate estimates of the true underlying values under certain confines. Some knowledge of the characteristics of the distributions of alternatives to be compared can assist in choosing which method will perform more consistently.

For utility functions of greater dimensions, as in the expo-power utility function, resampling procedures to estimate the confidence regions can have several drawbacks. The bootstrap method to estimate confidence regions has the limitation of being truncated at the sample extreme values as in single parameter cases. Confidence regions based on jackknife variance estimation can be highly variable as a result of the translation into multiple dimensions. In addition, as explained by Efron and Stein (1981), jackknife estimates of variance tend to be biased upward and thus are more conservative. The interplay between these aspects may act to result in overly wide confidence regions or biased or shifted regions. Methods to evaluate the influence of the particular sample data, such as jackknife-after-bootstrap procedures (Efron 1992), can assist in assessing the accuracy of the regions.

Of particular importance when testing for preference or estimating confidence regions over a range of the specific form of θ is the choice of grid size in choosing the discrete points where the statistic of interest will actually be evaluated. Depending on the particular utility function, the statistics as functions of θ can be curves, surfaces, or hyper surfaces. If the grid choice is not sufficiently small, specific ranges of θ may not be correctly estimated or represented and conclusions based on the given sample may be spurious, especially for utility functions that are not locally monotonic. Controlling experiment wide error rates, for example, with Bonferroni estimates of the confidence regions, is often a concern for this type of analysis. Since θ is not to be estimated, rather, it is taken as given, this type of analysis is more analogous to procedures which allow for new fixed terms to be taken as given once the parameters of interest have been estimated. It is therefore not recommended to use experiment wide adjustments in the confidence level based on the number of points in the grid space.

It has also been demonstrated in this section that, even when preferences are known based on explicit distributional and utility function specifications, a sample of data in these circumstances does not always yield the expected results when preferences are tested. Even under varying types of risk attitudes modeled by the respective utility functions, the results are highly similar from one assumption to the other. Based on Monte Carlo simulation experiments with varying sample sizes, the test statistic $T_N(\theta)$ appears to have the characteristics of asymptotic power that was highlighted in the previous chapter given increasing sample sizes and departures from the null hypothsis. An empirical example from unknown distributions will be the focus of the next chapter.

CHAPTER V

EMPIRICAL METHOD COMPARISON

5.1 Introduction

The previous chapter dealt with some of the more common assumptions involved in expected utility analysis. Given knowledge or an assumption about the underlying distribution of the data, explicit expressions can often be formulated for the sample certainty equivalent depending on the utility function. But, the distribution specifications for the sample certainty equivalent do not always have explicit analytical forms. Thus, even in these cases, alternative means must be used to assess the dispersion that arises from certainty equivalent estimation. Since the distributions of the original data were assumed to be known, the distribution of the test statistics could be approximated through Monte Carlo simulation. In a nonparametric setting, especially with small sample sizes, this is not a feasible option, so resampling methods can be employed to assess aspects of dispersion of the certainty equivalents.

This chapter will outline an example designed to compare a SERF preference ranking with some other methods. In keeping with some common notations in the expected utility literature (introduced in a simpler form in Von Neumann and Morgenstern (1953) in terms of comparing utilities), let $\succ_h$ indicate a binary preference ordering between two risky prospects, i.e. $X_1 \succ_h X_2$ indicates X_1 is preferred to X_2. In addition, $\succeq_h$ indicates "preferred to or equal" and $\sim_h$ indicates "indifference between" the two alternatives. Thus, $X_1 \succeq_h X_2$ and $X_2 \succeq_h X_1$ is equivalent to $X_1 \sim_h X_2$. The determination of the preference ordering is based on the preference criteria, e.g. mean-variance, stochastic dominance, stochastic dominance with respect to a function, etc. Thus, in the previous chapter,

$$X_2 \; SSD(k_\theta) \; X_1 \Leftrightarrow CE_2(\theta) > CE_1(\theta) \Leftrightarrow X_2 \succ_k X_1 \tag{5.1}$$

for a given θ, where $\succ_k$ is used to designate one alternative is "preferred with respect to the utility function k" over the other.

5.2 Empirical Example

The data in Table 1 will be used for some analyses. These are pseudo data contrived to highlight strengths and illustrate some of the drawbacks of some of the preference ranking methods and to detail the empirical methods in calculating preferences within the context of a SERF analysis. Noting the sample statistics, four of the series have equal means of \$100, one series has a mean of \$90, and the last has a mean of \$110. Two of each of the series have standard errors of \$25, \$30, and \$35, respectively. It is assumed that the sample is independent between observations. Where applicable, an initial wealth level of \$100 will be assumed.

Because of the small sample size and the distribution properties evidenced by some of the sample statistics, it would be difficult to assume a parametric distributional form for any or all of the alternatives. The sample exhibits varying levels of modality, skewness, and kurtosis. It should also be noted that there are dependencies between the samples, as is shown in the sample covariance matrix in Table 2. The dependency relationship can be an important factor in preference ranking since returns on investments are often functions of similar economic drivers. As illustrated in Figure 30, none of the sample distribution functions exhibit first degree dominance over any other alternatives.

Preference rankings using the mean-variance method are not altogether clear. Alternative X_j is preferred to alternative X_l in the sense of a mean-variance criterion

Table 1. Data and summary statistics for $n = 20$ observations of pseudo returns of $J = 6$ alternative investments.

Obs.	X_1	X_2	X_3	X_4	X_5	X_6
1	103.00	100.11	96.39	58.22	82.91	68.09
2	91.09	100.52	85.70	135.89	82.54	135.33
3	101.53	100.04	99.56	135.52	128.44	96.59
4	10.19	-1.62	81.52	45.50	52.85	166.09
5	104.72	116.11	61.66	57.67	65.42	89.61
6	95.35	99.84	78.45	107.21	102.74	167.76
7	110.42	100.22	113.17	82.21	114.72	103.42
8	92.62	109.42	77.19	144.52	59.71	140.95
9	94.03	98.68	101.97	133.94	89.73	84.30
10	96.77	98.32	73.66	138.18	82.02	133.83
11	88.64	96.60	110.02	64.47	66.57	96.61
12	99.02	117.54	109.97	71.51	82.03	95.82
13	110.04	177.08	72.16	60.69	87.36	57.39
14	191.34	102.04	161.46	105.22	140.57	41.52
15	104.78	100.42	118.06	83.45	70.67	115.77
16	106.91	101.75	96.23	59.17	113.66	98.13
17	94.64	89.07	136.83	133.70	93.75	150.73
18	104.11	94.32	131.03	139.95	117.08	137.00
19	101.08	100.04	114.32	124.25	112.03	86.36
20	99.70	99.49	80.68	118.75	55.22	134.71
Mean	100.00	100.00	100.00	100.00	90.00	110.00
St. Error	30.00	30.00	25.00	35.00	25.00	35.00
Minimum	10.19	-1.62	61.66	45.50	52.85	41.52
Maximum	191.34	177.08	161.46	144.52	140.57	167.76
Skewness	0.08	-1.26	0.71	-0.16	0.33	-0.08
Kurtosis	8.47	9.14	0.39	-1.73	-0.72	-0.64

($\succ_{mv}$) if it lies in the southeast quadrant of the X_l cross centered on the point based on the coordinates of the sample mean and standard deviation. Under this criteria,

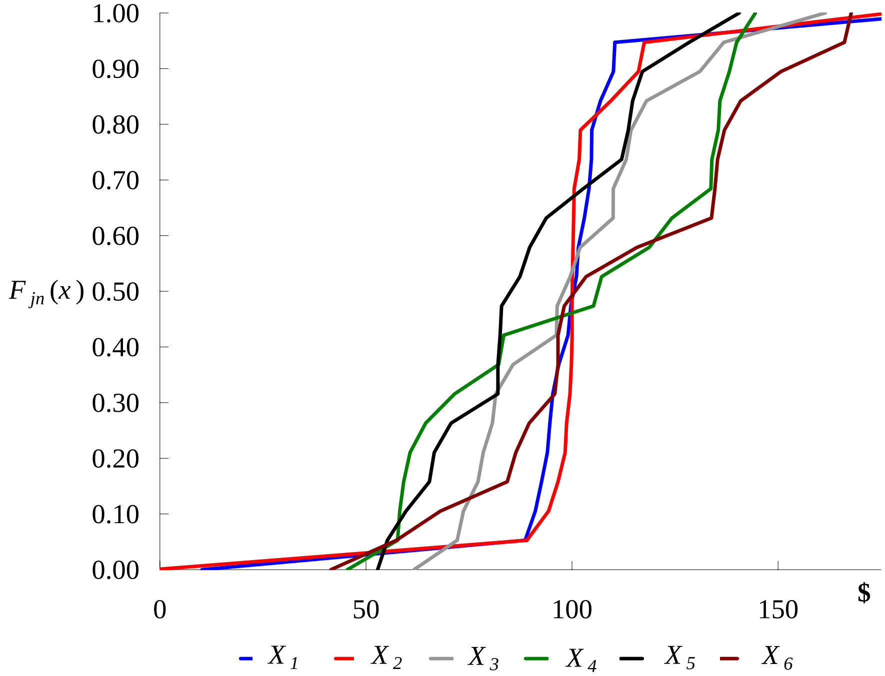


Figure 30. Linear-smoothed empirical distribution functions (edfs) for the returns of $J = 6$ alternative investments of sample size $n = 20$.

Table 2. Sample covariance matrix of the $J = 6$ alternatives.

	X_1	X_2	X_3	X_4	X_5	X_6
X_1	855.00	508.84	364.81	179.40	441.84	-640.30
X_2		855.02	-68.07	54.84	140.26	-538.74
X_3			593.76	150.21	354.53	-226.20
X_4				1,163.75	216.00	362.97
X_5					593.76	-316.19
X_6						1,163.73

it can be seen in the diagram in Figure 31, based on the sample estimates,

$$X_3 \succ_{mv} X_2 \sim_{mv} X_1 \succ_{mv} X_4. \tag{5.2}$$

Also note that $X_3 \succ_{mv} X_5$. Table 3 lists all of the preference orders for the six options. The table is read as "X_j, in the leftmost column, is preferred to, indifferent between, or subordinate to $X_{j+1}, X_{j+2}, \ldots$, listed in the top row, in terms of mean-variance preferences." Note that several of the comparisons are ambiguous. The mean-variance analysis does not indicate the preference trade-off for simultaneously increasing expected values and variability. Based on sample estimates of the first two moments, it is clear that only alternative X_4 can be eliminated from the efficient set of preferred alternatives. This type of mean-variance analysis does not account for dependency relationships between the variables.

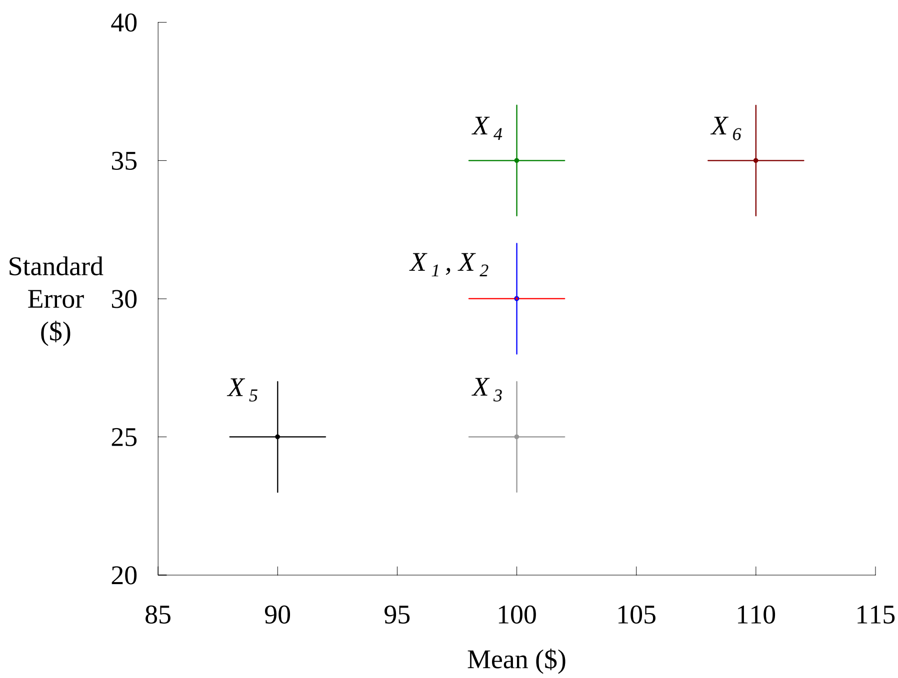


Figure 31. Mean-variance diagram for the returns of $J = 6$ alternative investments of sample size $n = 20$.

Assuming a power utility function, the SERF method was employed to compare the alternatives. Based on the sample, only three of the six alternatives could be attributed to the efficient set depending on the sub-set of θ, the relative risk aversion

Table 3. Preference orders based on a sample mean-variance analysis of the $J = 6$ alternatives.

	X_2	X_3	X_4	X_5	X_6
X_1	$\sim_{mv}$	$\prec_{mv}$	$\succ_{mv}$	?	?
X_2		$\prec_{mv}$	$\succ_{mv}$	?	?
X_3			$\succ_{mv}$	$\succ_{mv}$	?
X_4				?	$\prec_{mv}$
X_5					?

parameter. Figure 32 illustrates the certainty equivalent lines over a range of θ. For all risk averse agents ($\theta \in (-\infty, 0)$), only X_1 and X_6 would be preferred. X_1 would be preferred for those highly risk loving individuals, and X_6 would be preferred for those who are moderately risk loving to risk neutral. For all risk averse agents ($\theta \in (0, \infty)$), X_6, X_3, or both would be preferred. X_6 would be preferred by risk neutral to moderately risk averse agents, and those who are highly risk averse would prefer alternative X_3.

Pair wise comparisons can be made in the context of a SERF analysis given a specified utility function parameterization. For $\theta = 0$, or risk neutral agents, X_6 is preferred over the remaining alternatives and X_5 is dominated by all other alternatives. A risk neutral agent would be indifferent between the remaining four alternatives. Over the entire range of θ, there are several preference changes, which illustrates why comparisons at discrete values of θ can be misleading when allocating alternatives to the efficient set. For example, numerically evaluating the preferences at only $\theta = -15$ and $\theta = 15$ would exclude X_6 from the efficient set even though it appears to dominate the other alternatives in a sub-set of this range.

The certainty equivalent is a means to summarize the distribution of a random

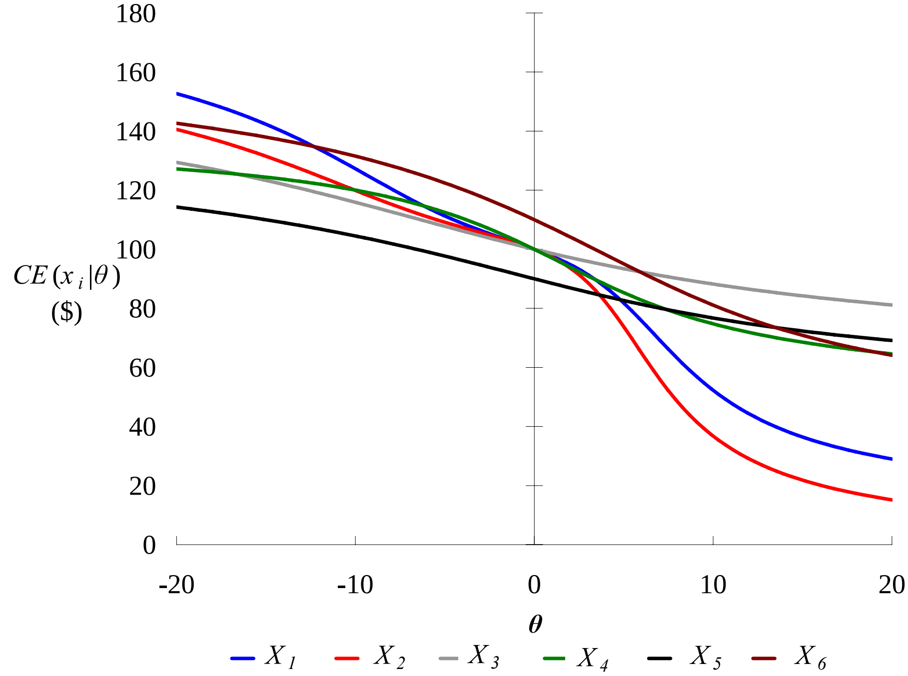


Figure 32. *Stochastic efficiency with respect to a function (SERF) analysis for the returns of $J = 6$ alternative investments of sample size $n = 20$. θ, an index of relative risk aversion, is the parameter of the power utility function used in formulating the certainty equivalents.*

variable and compare it to those of other distributions by means of a static value. The true certainty equivalent is a static value, but since the sample certainty equivalent is also a random variable, it is proper to examine preference rankings in terms of relevant confidence regions for the statistic. As was presented previously, confidence regions overlap a great deal for alternatives with nearly equivalent means and variances.

This is the case in the present example. Figure 33 illustrates the estimated confidence region with the smallest width, that for X_5, with shaded vertical lines. If it were assumed that the certainty equivalents for each of the remaining alternatives were known (and independent of X_5), it could only be concluded, in addition to being generally dominated over $\theta \leq 0$, based on the sample estimates, that X_5 dominates series X_1 and X_2 over a sub-set of highly risk averse agents. Since the other alternatives are random and dependent, even the previous conclusion is somewhat suspect. Thus, in cases where the certainty equivalent surface is contained in the confidence region of one or more alternatives, concluding that there is a preference ranking between these alternatives given a specified level of confidence may be suspect.

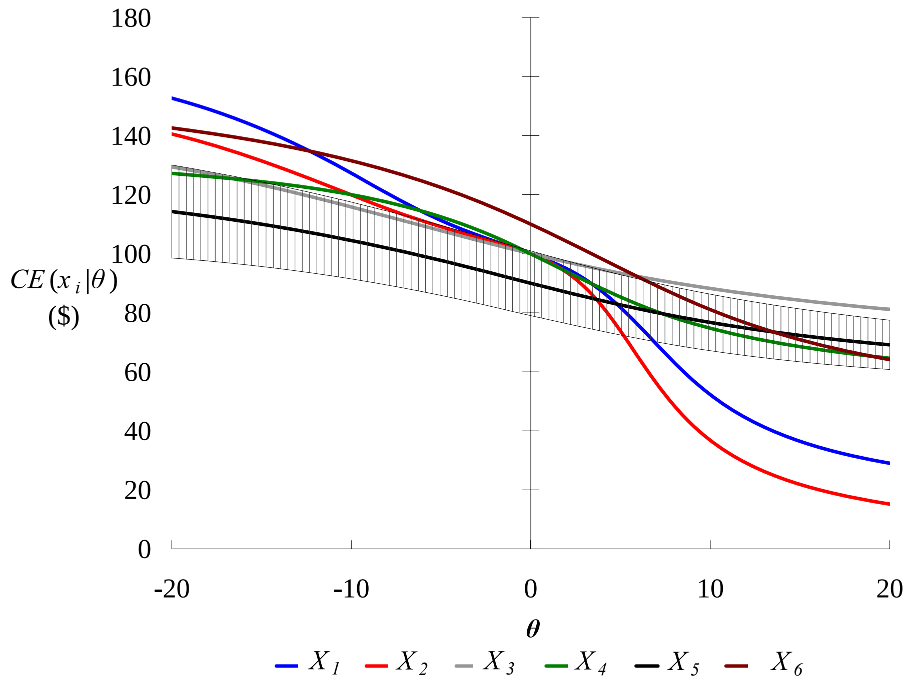


Figure 33. Stochastic efficiency with respect to a function (SERF) analysis for the returns of $J = 6$ alternative investments of sample size $n = 20$. θ, an index of relative risk aversion, is the parameter of the power utility function used in formulating the certainty equivalents. An approximate 95% confidence regions based on jackknife variance estimation procedures is given for Alternative 5.

These types of assessments lead to the necessity of using the formal testing procedure as developed in Chapter III. In addition to showing the direction of preference, the testing method also indicates magnitudes of preference based on the given sample and confidence level. Noting that from the mean-variance and SERF analyses, Alternative 5 appears to be at the lower end of preferences. This alternative will be used as a base for pair wise hypothesis tests of second degree stochastic dominance with respect to a function. Figure 34 shows the results of the pair wise tests of the hypothesis $H_1 : X_j \ SSD(k) \ X_5$ for $j \neq 5$ over the set where $\theta \in \Theta_1 = (-20, 20)$ using a 95% confidence level. Each test, if conducted independently, rejects the null hypothesis of indifference if $T_{40}(\theta) < z_\alpha \approx -1.645$. From this illustration, it can be seen that only Alternatives 1, 2, or 3 dominate Alternative 5 with respect to the power utility function depending on the subset of θ.

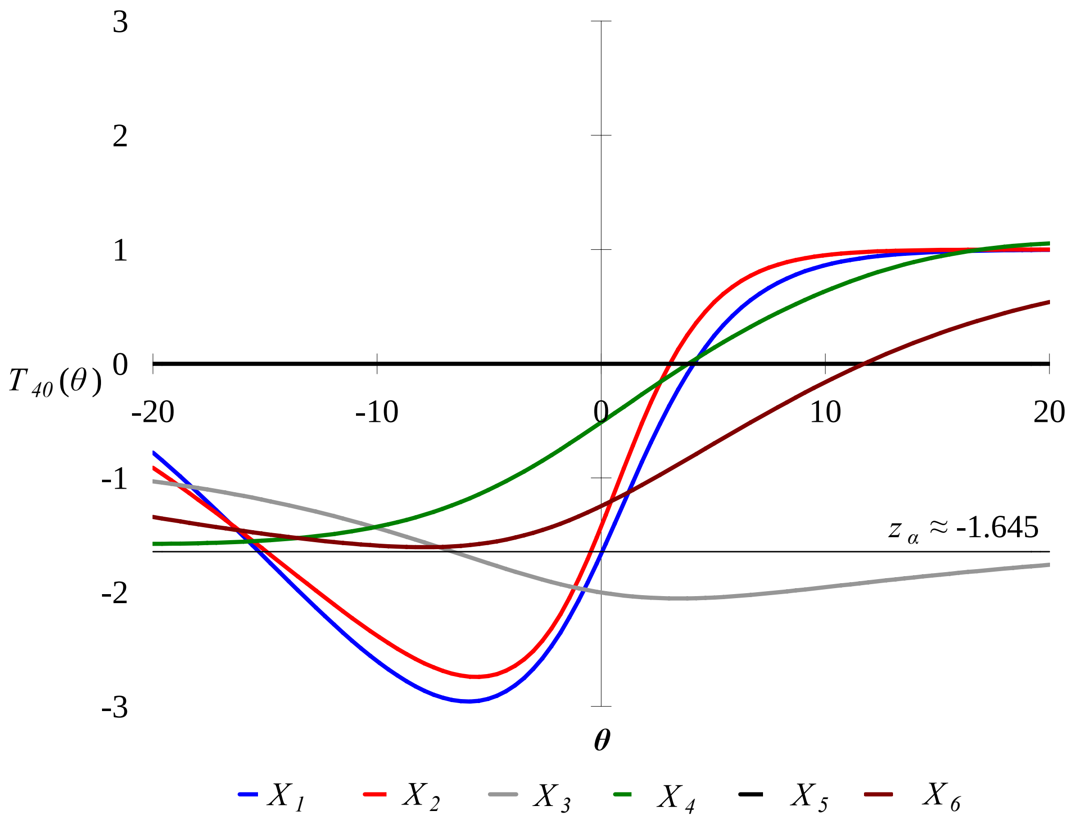


Figure 34. *Test statistic,* $T_{40}(\theta)$*, based on sample of* $n = 20$ *returns of* $J = 6$ *alternative investments,*$X_1, \ldots, X_6$*. The null hypothesis of indifference between two given alternatives is rejected in favor of* $H_1 : X_j \ SSD(k_\theta) \ X_5, j \neq 5$ *in large samples for values of* $T_{40}(\theta) < z_{\alpha=0.05}$*, where* k_θ *is the power utility function with* θ *as the coefficient of relative risk aversion.*

Comparing the results to the pair wise comparisons in the mean-variance analysis in Table 3, it can be seen that there appears to exist ranges of risk preferences where the preference ordering are neither indifferent nor indeterminate based on the sample. It can also be seen that strict preference rankings resulting from the mean-variance are not everywhere statistically significant over the given range of θ. Where comparisons to X_5 lead to largely indeterminate results in the mean-variance analysis, a sense of the direction of preference relative to this alternative can be gained by whether or not a specified statistic value is negative. Since the jackknife method was employed to estimate the variance of the test statistics at given values of θ, the between variable dependencies have been maintained and reflected in the results.

Let $\succ_k$, $\sim_k$, and $\prec_k$ denote "dominates", "is indifferent to", and "is dominated by" with respect to a utility function k, respectively, where k is the power utility function in this example. When, in this case, $\theta = 0$, the test reduces to the form in ESY, i.e., $H_1 : X_j\ SSD\ X_5$ for $j \neq 5$. Table 4 presents the results of these tests for pair wise comparisons. The tests reveal indifference in general, apart from X_5 being second degree dominated by X_1 and X_3.

Table 4. Preference orders based on pair wise tests of second degree stochastic dominance of the $J = 6$ alternatives from unknown distributions based on a sample size of $n = 20$ and 95% confidence level.

$\theta = 0$	X_2	X_3	X_4	X_5	X_6
X_1	$\sim_k$	$\sim_k$	$\sim_k$	$\succ_k$	$\sim_k$
X_2		$\sim_k$	$\sim_k$	$\sim_k$	$\sim_k$
X_3			$\sim_k$	$\succ_k$	$\sim_k$
X_4				$\sim_k$	$\sim_k$
X_5					$\sim_k$

Similar to Meyer's method (1977b), let the arbitrary range of $\theta \in [-6, 6]$, representing moderately risk loving to moderately risk averse agents, be a range of interest. Table 5 represents the preference ordering for the outer bounds of this range. For these values of θ, the preference of X_3 over X_5 resulting from the mean-variance analysis is confirmed. Again, statistical indifference between two alternatives provides more information than the indeterminate results presented in the mean-variance analysis. Also, each of the results, although not the same, slightly overlaps those of the test of second degree dominance.

Table 5. Preference orders based on a sample second degree stochastic dominance with respect to a function analysis of $J = 6$ alternatives from unknown distributions based on a sample size of $n = 20$ and 95% confidence level. k is specified to be the power utility function. The pair wise preference orderings are based on $\theta = -6$ and $\theta = 6$, where θ is the coefficient of relative risk aversion.

$\theta = -6$	X_2	X_3	X_4	X_5	X_6
X_1	$\sim_k$	$\sim_k$	$\sim_k$	$\succ_k$	$\sim_k$
X_2		$\sim_k$	$\sim_k$	$\succ_k$	$\sim_k$
X_3			$\sim_k$	$\succ_k$	$\sim_k$
X_4				$\sim_k$	$\sim_k$
X_5					$\sim_k$

$\theta = 6$	X_2	X_3	X_4	X_5	X_6
X_1	$\sim_k$	$\sim_k$	$\sim_k$	$\sim_k$	$\sim_k$
X_2		$\sim_k$	$\sim_k$	$\sim_k$	$\sim_k$
X_3			$\succ_k$	$\succ_k$	$\sim_k$
X_4				$\sim_k$	$\sim_k$
X_5					$\sim_k$

5.3 Conclusions

The purpose of this chapter was to demonstrate some methods of preference ordering for risky alternatives in the context of an empirical example. For the particular example, there was no first degree dominance of one alternative over another. A few examples of second tier methods of comparison were presented, with some similar and some dissimilar results. The similarities came largely from the fact that the first two moments of the samples were the same or similar, The differences were primarily a result of different underlying distributions from which the samples came, the properties of the individual samples, specifically extreme values, and the specific assumptions and criteria of the ranking methods.

Methods using stochastic dominance with respect to a function assumptions do not rely strictly on the first moments of random variables but rather on a weighted comparison of the entire distribution represented by the sample. Depending on the utility function, specifications of θ allow for weighting toward sample extreme values or the center of the distribution. Because of its flexibility, this method subsumes many of the other preference ranking methods, depending on the utility function and specific parameterization[1].

Regarding stochastic dominance with respect to a function methods, risk preference ranges are typically restricted to moderately risk loving to moderately risk averse ranges, where special emphasis is put on risk averse agents because of their particular operations in the marketplace. For this example, assuming moderate risk aversion, X_3 is preferred to X_4 and X_5. These particular results are the same as the mean-variance preferences, which might lend some insight into the properties of the distributions of these three alternatives.

As with other testing procedures attempting to evaluate properties of distributions, e.g., goodness of fit tests, the results of ranking methods may differ depending on the initial assumptions implicit in the procedure. First degree stochastic dominance depends on the entire distribution of the alternatives to be compared. Establishing these criteria implies many of the others. Mean-variance type analyses depend on the first two moments of the alternatives, but rankings based on this procedure

[1]Use of the quadratic utility function in expected utility analyses, especially in conjunction with distributions fully specified by their first two moments, yields mean-variance analyses. The common criticism of using the quadratic utility is that it exhibits increasing absolute risk aversion.

may not imply stochastic dominance of one alternative over another. Methods evaluating the certainty equivalents of alternatives such as stochastic efficiency with respect to a function produce rankings over ranges of risk indices but may also result in over confident assertions if the respective variances are ignored. Testing for stochastic dominance with respect to a function attempts to take the agents decision making mechanism into account when ranking preferences. This method may correspond to mean-variance type analyses depending on the nature of the true distributions of the alternatives of interest and the utility function assumed.

CHAPTER VI

SUMMARY AND FURTHER STUDY

6.1 Summary

The goal of this dissertation has been to extend the methodology of risk analysis in the context of expected utility theory. The problem framework involves two or more risky investment alternatives, typically measured in monetary units or rates of returns, which come from unknown distributions. It is assumed that a sample of the alternatives exists and that the distributions can be estimated, e.g., with empirical distribution functions. Determination of a preference ranking between the alternatives based on an investment agent's decision-making mechanism, specifically utility, is the ultimate purpose of the analysis.

The first objective of this dissertation has been comparing the certainty equivalents of the alternatives. This involves ranking the mean values given a parameterization of the assumed utility function; but, it also necessitates estimating the variance of the certainty equivalent values. It has been demonstrated that the estimation of the variance of the certainty equivalent is often a nontrivial undertaking even when the underlying distributions is known. Two methods have been presented to attempt to estimate dispersion in a nonparametric setting: bootstrap resampling of the original sample to estimate confidence regions based on quantiles from the bootstrap distribution and jackknife variance estimation procedures. Some of the advantages and disadvantages of each method have been discussed.

The second objective of this dissertation has been the development of a nonparametric asymptotic test of second degree stochastic dominance with respect to a function. This test procedure is an application of the ranking methods presented by

Meyer (1977b) through a generalization of the test of second degree stochastic dominance outlined in Eubank, Schechtman, and Yitzhaki (1993). The generalized test has been shown to have a representation as the Eubank, et. al. (1993) test applied to the transformations of the sample data from two or more risky alternatives into utility measures. The generalized test can be presented as a function of the parameterization of the assumed utility function, and thus can be conducted in the context of a decision-making agent's specific ranges of risk aversion.

The new methodology developed in this study have been applied to examples of data from known distributions with a few specific utility function assumptions. The properties of both the confidence regions produced by the dispersion estimation procedures and the test statistics as a function of specific utility function parameterizations were highlighted in these cases for finite samples. An empirical example has been given to demonstrate varying dominance methods and how they relate.

6.2 Further Study

To present the general characteristics of the statistics related to each objective, some assumptions were made about the sample data. For future research, evaluation of the performance of the ranking methods might be conducted in forecasting scenarios, specifically, when modeling of a forecasted investment value. An ideal procedure to incorporate these methods would be comparing alternative investments to a relatively risk-free investment, e.g., a bond rate or certificate of deposit. The analysis would then be conducted in terms of the risk premia above the risk-free rate. This type of analysis would necessitate fully incorporating the dependency relationships between the alternatives for confidence region estimation and hypothesis testing. The framework would then most likely be a non-i.i.d. multivariate time series model.

Another distinct continuance of this research would involve a Baysian framework of the testing procedure. In this dissertation, no effort has been made to estimate the utility function parameter. If hypotheses are implemented across relevant values of a risk aversion index, the actual preference made by the agent can be used to update the feasible range or distribution of the risk aversion index for subsequent hypotheses. Inferences on the distribution of the utility parameter can be made through posterior estimation given the actual decision. This will be a particular worthwhile approach to consider for time-dependent utility functions to evaluate changes in preferences.

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APPENDIX A

PROOFS AND DERIVATIONS

Proof of Theorem 1

The following will show the result of Theorem 1 that

$$\begin{aligned} D_{n,m}(\theta) &= \frac{1}{2}\left[\int_{-\infty}^{\infty} d_{n,m}(x|\theta)dG_m(x) + \int_{-\infty}^{\infty} d_{n,m}(y|\theta)dF_n(y)\right] \\ &= \frac{1}{2}\left[\frac{1}{n}\sum_{i=1}^{n} W_i - \frac{1}{m}\sum_{j=1}^{m} T_j\right] \\ &+ \frac{1}{2}\left[\int_{-\infty}^{\infty} F_n(x)\left[1-F_n(x)\right]dk(x|\theta) - \int_{-\infty}^{\infty} G_m(y)\left[1-G_m(y)\right]dk(y|\theta)\right], \end{aligned}$$

where $W_i = \int_{x_i}^{\infty} [k(t|\theta) - k(x_i|\theta)]dG_m(t)$ and $T_j = \int_{y_j}^{\infty} [k(t|\theta) - k(y_j|\theta)]dF_n(t)$.

Proof. For the first component of $D_{n,m}(\theta)$, we show that

$$\begin{aligned} \int_{-\infty}^{\infty} d_{n,m}(x|\theta)dG_m(x) &= \int_{-\infty}^{\infty}\int_{-\infty}^{x} \left[F_n(t) - G_m(t)\right]dk(t|\theta)dG_m(x) \\ &= \frac{1}{n}\sum_{i=1}^{n} W_i - \int_{-\infty}^{\infty} \left[1-G_m(t)\right]G_m(t)dk(t|\theta), \end{aligned}$$

where $W_i = \int_{x_i}^{\infty} [k(t|\theta) - k(x_i|\theta)]dG_m(t)$. A similar result will then hold for the second component of $D_{n,m}(\theta)$.

Now, explicitly, we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} d_{n,m}(x|\theta)dG_m(x) \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{x} \left[F_n(t) - G_m(t)\right] dk(t|\theta)dG_m(x) \\
&= \int_{-\infty}^{\infty}\left[\int_{t}^{\infty} dG_m(x)\right] \left[F_n(t) - G_m(t)\right] dk(t|\theta) \\
&= \int_{-\infty}^{\infty} \left[1 - G_m(t)\right] \left[F_n(t) - G_m(t)\right] dk(t|\theta) \\
&= \int_{-\infty}^{\infty} \left[1 - G_m(t)\right] F_n(t)dk(t|\theta) - \int_{-\infty}^{\infty} \left[1 - G_m(t)\right] G_m(t)dk(t|\theta).
\end{aligned}$$

It then must be shown that

$$\int_{-\infty}^{\infty} \left[1 - G_m(t)\right] F_n(t)dk(t|\theta) = \frac{1}{n}\sum_{i=1}^{n}\int_{x_i}^{\infty} \left[k(t|\theta) - k(x_i|\theta)\right] dG_m(t) = \frac{1}{n}\sum_{i=1}^{n} W_i.$$

Now let $x_{(1)} < x_{(2)} < \cdots < x_{(n)}$ be the ordered x observations and let $I_{[a,b]}(t)$ be the indicator function that takes on a value of one for values in $[a, b]$ and zero otherwise. Then $F_n(t) = \frac{1}{n}\sum_{i=1}^{n} I_{[x_{(i)},\infty]}(t)$, and the first term in our previous expression becomes

$$\begin{aligned}
\int_{-\infty}^{\infty} \left[1 - G_m(t)\right] F_n(t)dk(t|\theta) &= \frac{1}{n}\sum_{i=1}^{n}\int_{-\infty}^{\infty} \left[1 - G_m(t)\right] I_{[x_{(i)},\infty]}(t)dk(t|\theta) \\
&= \frac{1}{n}\sum_{i=1}^{n}\int_{x_i}^{\infty} \left[1 - G_m(t)\right] dk(t|\theta)\ .
\end{aligned}$$

Integration by parts produces

$$\begin{aligned}
&= \frac{1}{n}\sum_{i=1}^{n}\left[-k(x_i|\theta)(1 - G_m(x_i)) + \int_{x_i}^{\infty} k(t|\theta)dG_m(t)\right] \\
&= \frac{1}{n}\sum_{i=1}^{n}\int_{x_i}^{\infty} \left[k(t|\theta) - k(x_i|\theta)\right] dG_m(t) = \frac{1}{n}\sum_{i=1}^{n} W_i.
\end{aligned}$$

□

which is in accordance with the previous definition for W_i. A similar process yields $\frac{1}{m}\sum_{j=1}^{m} T_j$ for the complementary component of $D_{n,m}(\theta)$.

Proof of Theorem 2

Now, recall the result of Theorem 2:

$$D_{n,m}(\theta) = \frac{1}{2}\left\{\overline{k_\theta(Y)} - \overline{k_\theta(X)} + \frac{1}{2}GMD[k_\theta(X)] - \frac{1}{2}GMD[k_\theta(Y)]\right\} .$$

Proof. The proof proceeds by first showing that

$$\frac{1}{n}\sum_{i=1}^{n} W_i - \frac{1}{m}\sum_{j=1}^{m} T_j = \overline{k_\theta(Y)} - \overline{k_\theta(X)}.$$

Note that, from the previous proof,

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n} W_i &= \int_{-\infty}^{\infty} [1 - G_m(t)]\, F_n(t) dk(t|\theta) \\ &= \frac{1}{m}\sum_{j=1}^{m}\int_{-\infty}^{y_i} F_n(t) dk(t|\theta),\end{aligned}$$

and by integration by parts we find that

$$\begin{aligned}\frac{1}{m}\sum_{j=1}^{m}\int_{-\infty}^{y_i} F_n(t) dk(t|\theta) &= \frac{1}{m}\sum_{j=1}^{m}\left[k(y_j|\theta)F_n(y_j) - \int_{-\infty}^{y_j} k(t|\theta) dF_n(t)\right] \\ &= \frac{1}{m}\sum_{j=1}^{m}\int_{-\infty}^{y_j} [k(y_j|\theta) - k(t|\theta)]\, dF_n(t) = \frac{1}{n}\sum_{i=1}^{n} W_i.\end{aligned}$$

Recall also that

$$T_j = \int_{y_j}^{\infty} [k(t|\theta) - k(y_j|\theta)] dF_n(t)$$

and so

$$\frac{1}{m}\sum_{j=1}^{m} T_j = \frac{1}{m}\sum_{j=1}^{m}\int_{y_j}^{\infty} [k(t|\theta) - k(y_j|\theta)] dF_n(t).$$

Therefore,

$$
\begin{aligned}
& \frac{1}{n}\sum_{i=1}^{n} W_i - \frac{1}{m}\sum_{j=1}^{m} T_j \\
= \; & \frac{1}{m}\sum_{j=1}^{m}\left\{\int_{-\infty}^{y_j}\left[k(y_j|\theta) - k(t|\theta)\right] dF_n(t) - \int_{y_j}^{\infty}\left[k(t|\theta) - k(y_j|\theta)\right] dF_n(t)\right\} \\
= \; & \frac{1}{m}\sum_{j=1}^{m}\left[\int_{-\infty}^{y_j} k(y_j|\theta) dF_n(t) - \int_{-\infty}^{y_j} k(t|\theta) dF_n(t)\right. \\
& \qquad \left. - \int_{y_j}^{\infty} k(t|\theta) dF_n(t) + \int_{y_j}^{\infty} k(y_j|\theta) dF_n(t)\right] \\
= \; & \frac{1}{m}\sum_{j=1}^{m}\left[k(y_j|\theta) - \int_{-\infty}^{\infty} k(t|\theta) dF_n(t)\right] \\
= \; & \frac{1}{m}\sum_{j=1}^{m} k(y_j|\theta) - \frac{1}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\frac{1}{n} k(x_i|\theta) \\
= \; & \overline{k_\theta(Y)} - \overline{k_\theta(X)}.
\end{aligned}
$$

Now, with respect to the second term of $\int_{-\infty}^{\infty} d_{n,m}(x|\theta) dG_m(x)$, that being

$$
\int_{-\infty}^{\infty}\left[1 - G_m(t)\right] G_m(t) dk(t|\theta),
$$

a change of variable, $v_\theta = k(t)$, results in

$$
\int_V \left[1 - G_m(k^{-1}(v|\theta))\right] G_m(k^{-1}(v|\theta)) dv,
$$

assuming $k^{-1}(\cdot)$ exists. And thus, reviewing Kendall and Stuart (Chapter 2, 1977) and Yitzhaki (1982), the term becomes

$$
\frac{1}{2\binom{m}{2}} \sum\sum_{i<j} \left| k(Y_i|\theta) - k(Y_j|\theta) \right| ,
$$

or $\frac{1}{2} GMD[k_\theta(Y)]$. Again, a similar result holds for $\int_{-\infty}^{\infty}\left[1 - F_n(t)\right] F_n(t) dk(t|\theta)$, and the theorem has been proved. $\square$

Derivation of Large Sample Mean of the Test Statistic $D_{n,m}(\theta)$

From ESY, the large sample mean of the test statistic $D_{n,m}$ is given as

$$-2\int(1-F(t))\delta(t)dt$$

with respect to a random variable X with distribution function $F(\cdot)$. As a generalization, with the random variables $V = k(X|\theta)$ and $W = k(Y|\theta)$, where $k(\cdot)$ is a utility function, let $F_n(\cdot)$ be the empirical distribution function (EDF) of a sample of n random variables V and $G_m(\cdot)$ be the EDF of a sample of m random variables W. Letting $N = n + m$, use the approximation for the function δ

$$\hat{\delta}_N(x) = \frac{G_m(x) - F_n(x)}{\sqrt{N}}$$

to estimate

$$\begin{aligned}
\hat{\mu}_{D_{n,m}(\theta)} &= -2\int[1-F_n(t)]\delta_N(t)dt \\
&= \frac{-2}{\sqrt{N}}\int[1-F_n(t)][G_m(t)-F_n(t)]dt \\
&= \frac{-2}{\sqrt{N}}\int[1-F_n(t)]G_m(t)-[1-F_n(t)]F_n(t)dt.
\end{aligned}$$

Recall that $\int[1-F_n(t)]F_n(t)dt = \frac{1}{2}GMD(V)$, or Gini's mean difference of the sample from V. Similar to the previous proofs, we have that

$$\int[1-F_n(t)]G_m(t)dt = \frac{1}{n}\sum_{i=1}^{n}\int_{-\infty}^{v_i}G_m(t)dt.$$

Focusing on the integral, we have

$$\int_{-\infty}^{v_i}G_m(t)dt = \frac{1}{m}\sum_{j=1}^{m}(v_i - w_{(j)})I_{(-\infty,v_i]}(w_{(j)})$$

where $w_{(j)}$ is the j^{th} order statistic and $I(\cdot)$ is an indicator function. Summing the integrals, we get

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n}\int_{-\infty}^{v_i} G_m(t)dt &= \frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m}(v_i - w_{(j)})I_{(-\infty,v_i]}(w_{(j)}) \\ &= \frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m}(v_i - w_j)I_{(w_j<v_i)}\end{aligned}$$

and therefore

$$\hat{\mu}_{D_{n,m}(\theta)} = \frac{-2}{\sqrt{N}}\left[\frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m}(v_i - w_j)I_{(w_j<v_i)} - \frac{1}{2}GMD(v)\right]$$

or

$$\hat{\mu}_{D_{n,m}(\theta)} = \frac{-2}{\sqrt{N}}\left[\frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m}[k_\theta(x_i) - k_\theta(y_j)]I_{[k_\theta(x_j)<k_\theta(v_i)]} - \frac{1}{2}GMD[k_\theta(X)]\right] .$$

APPENDIX B

SAMPLE DATA

Table 6. Sample data used in Chapter IV. Data are two independent samples from normal distributions with means $\mu_1 = 100$ and $\mu_2 = 100$ and variances $\sigma_1^2 = 30^2$ and $\sigma_2^2 = 25^2$, respectively.

Obs.	X_1	X_2	Obs.	X_1	X_2
1	82.903	91.158	16	108.836	100.944
2	123.048	108.281	17	53.285	130.157
3	69.455	102.274	18	97.227	83.183
4	153.012	110.345	19	81.653	83.321
5	105.831	148.287	20	109.146	84.334
6	123.602	56.149	21	39.544	126.933
7	68.345	93.571	22	97.293	155.465
8	89.666	149.767	23	88.083	111.394
9	103.375	122.938	24	138.936	83.651
10	99.321	111.277	25	42.471	94.015
11	98.461	77.727	26	110.503	102.208
12	66.590	119.332	27	200.371	88.012
13	85.649	97.488	28	113.735	90.647
14	76.005	67.052	29	161.283	88.018
15	157.699	140.987	30	62.741	48.360

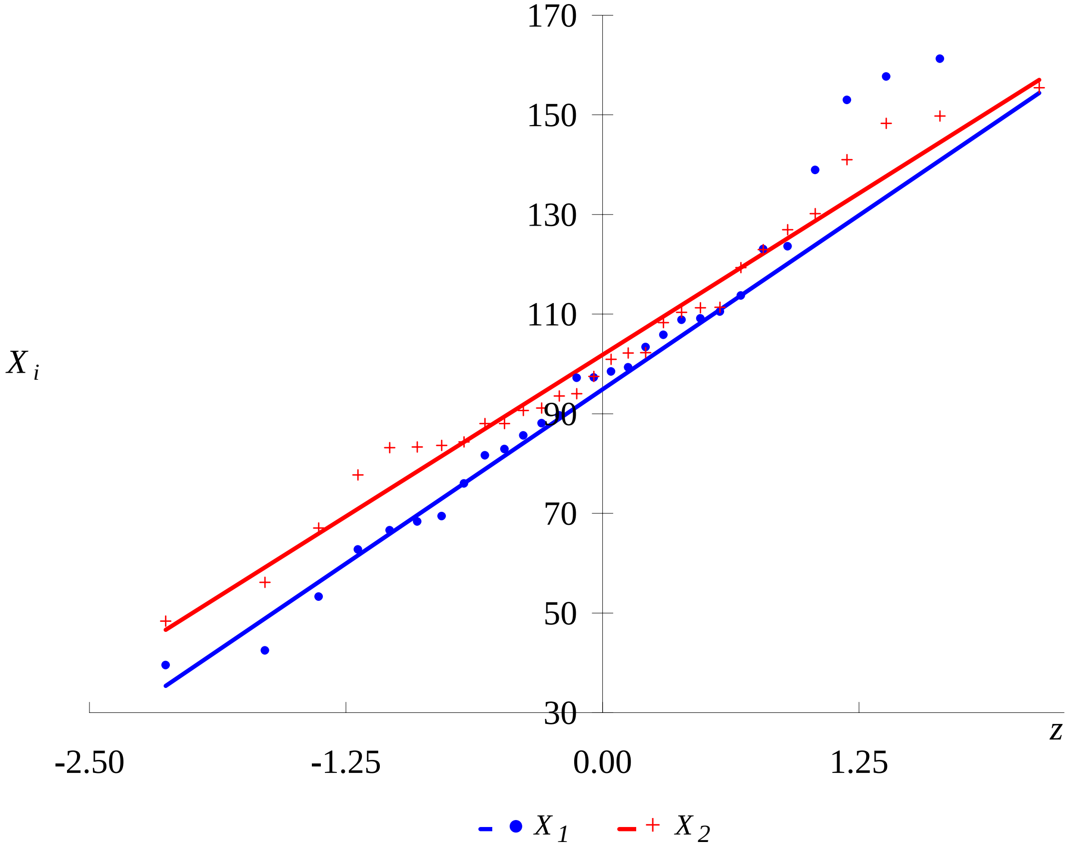

Figure 35. Normal probability plot of standard normal quantiles (z) versus samples of size $n = 30$ from two random variables (X_1 and X_2). Data are from independent normal distributions with means $\mu_1 = 100$ and $\mu_2 = 100$ and variances $\sigma_1^2 = 30^2$ and $\sigma_2^2 = 25^2$, respectively.

VITA

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